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RIESZ BASIS PROPERTY AND RELATED RESULTS FOR A RAO-NAKRA SANDWICH BEAM

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Abstract. We consider a three layer Rao-Nakra sandwich beam with distinct wave speeds. We prove that the eigenvectors form a Riesz basis for the natural energy space. In the damped case, we give precise conditions under which there is a uniform exponential decay of energy. We also consider the problem of boundary control using bending moment and lateral force control at one end. We prove that the space of exact controllability has finite co-dimension and provide sufficient conditions (related to small damping) for exact controllability to a zero energy state.

1. **Introduction.** The classical Rao-Nakra [8] sandwich beam model consists of two outer "face plate" layers (which are assumed to be relatively stiff) which "sandwich" a much more compliant "core layer". The model is derived using Euler-Bernoulli beam assumptions for the face plate layers, Timoshenko beam assumptions for the core layer and a "no-slip" assumption for the displacements along the interface. The following formulation of the Rao-Nakra model (adapted here for a beam model) is described in Hansen[2].

$$\begin{array}{rcll} m\ddot{w} - \alpha D_x^2\ddot{w} + KD_x^4 w - D_x N h_2(G_2\varphi + \tilde{G}_2\dot{\varphi}) & = & 0 & \text{on } (0,L)\times(0,\infty) \\ \mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\ddot{\mathbf{v}}_{\mathcal{O}} - \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}D_x^2\mathbf{v}_{\mathcal{O}} + \mathbf{B}^{\mathbf{T}}(G_2\varphi + \tilde{G}_2\dot{\varphi}) & = & 0 & \text{on } (0,L)\times(0,\infty) \end{array} \tag{1}$$

where $\varphi = h_2^{-1}\mathbf{B}v_{\mathcal{O}} + Nw_x$. In addition we consider the following controlled boundary conditions:

$$\begin{array}{rcll} w(0,t) = D_x^2 w(0,t) = D_x\mathbf{v}_{\mathcal{O}}(0,t) = w(L,t) & = & 0 & t>0, \\ KD_x^2 w(L,t) = M(t),\ \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}D_x\mathbf{v}_{\mathcal{O}}(L,t) = \mathbf{g}_{\mathcal{O}}(t) & & & t>0 \end{array}. \tag{2}$$

In the above, w denotes the transverse displacement of the beam, φ denotes the shear angle of the core layer, $\mathbf{v}_{\mathcal{O}} = (v_1, v_3)^T$ is the vector of longitudinal displacement along the neutral axis of the outer layers. ($i = 1, 3$ is for the outer layers, $i = 2$ is for the core layer.) The density of the ith layer is denoted ρ_i, the thickness h_i, the Young's modulus E_i, the shear modulus of the core layer is G_2. We let $m = \sum h_i\rho_i$ denote the mass density per length, $\alpha = \rho_1 h_1^3/12 + \rho_3 h_3^3/12$ denote a moment of inertia parameter, $K = E_1 h_1^3/12 + E_3 h_3^3/12$ denote the bending stiffness. In addition,

$$\mathbf{p}_{\mathcal{O}} = \operatorname{diag}(\rho_1, \rho_3), \quad \mathbf{h}_{\mathcal{O}} = \operatorname{diag}(h_1, h_3), \quad \mathbf{E}_{\mathcal{O}} = \operatorname{diag}(E_1, E_3)$$
$$\mathbf{B} = (-1, 1), \qquad N = \frac{h_1 + h_2 + h_3}{h_2}.$$

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The boundary control functions acting at the right end of the beam are $M(t)$, the applied moment, and $\mathbf{g}_{\mathcal{O}}(t) = (g_1(t), g_3(t))^T$, the longitudinal force. (See Hansen[2] for a precise definition of the applied forces.) Our main result is the following:

Theorem 1. *Assume the wave speeds* $\sqrt{\frac{K}{\alpha}}, \sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}$ *are distinct. Then the eigenvectors form a Riesz basis for the finite energy space* $X_0 \times X_1$.

We also prove several consequences of Theorem 1. First, under some minor (and necessary) restrictions on the the parameters, if the damping $\tilde{G}_2$ is positive, the energy of solutions of the homogeneous problem decay to zero at a uniform exponential rate. (See Proposition 2, Corollary 1).

Secondly, we consider the problem of controlling an initial finite energy state to another in time T with controls $\mathbf{g}_{\mathcal{O}}$, $M(t)$ belonging to $L^2(0,T)$. We prove that if $T > \tau$ where

$$\tau = 2L\left[\min\left(\sqrt{\frac{K}{\alpha}}, \sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}\right)\right]^{-1} \tag{3}$$

Then the system (1)–(2) is exactly controllable modulo a finite dimensional quotient. If the coupling terms G_2 and $\tilde{G}_2$ are sufficiently small, this quotient consists of the the space determined by "zero energy" uncontrollable state $w = 0, (v_1, v_3) = (1, 1)$.

Remark 1. Throughout this paper we restrict our analysis to the case of distinct wave speeds, as in Theorem 1.

The paper is organized as follows. In Section 2 we describe the semigroup formulation of (1) with $M(t) = 0, \mathbf{g}_{\mathcal{O}} = 0$. In Section 3 we prove Theorem 1. In Section 4 we prove the controllability results.

2. **Semigroup formulation.** Let $U = (u, \mathbf{u})^T := (w, \mathbf{v}_{\mathcal{O}})^T$, $V = (v, \mathbf{v})^T := (\dot{w}, \dot{\mathbf{v}}_{\mathcal{O}})^T$, $Y = (U, V)$. Also define $J : H^2(0,L) \cap H_0^1(0,L) \to L^2(0,L)$ by $J\theta = m\theta - \alpha D_x^2\theta$. The first order form of (1) with M and $\mathbf{g}_{\mathcal{O}}$ set to zero is

$$\frac{dY}{dt} = \mathcal{A}Y := \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix},$$

$$\text{where } A_1U = \begin{pmatrix} J^{-1}(-KD_x^4u + D_xNh_2G_2[h_2^{-1}(\mathbf{Bu} + h_2ND_xu)]) \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}[\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}D_x^2\mathbf{u} - \mathbf{B}^TG_2[h_2^{-1}(\mathbf{Bu} + h_2ND_xu)]] \end{pmatrix}$$

$$A_2V = \begin{pmatrix} J^{-1}(D_xNh_2\tilde{G}_2[h_2^{-1}(\mathbf{Bv} + h_2ND_xv)] \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}[-\mathbf{B}^T\tilde{G}_2[h_2^{-1}(\mathbf{Bv} + h_2ND_xv)]]) \end{pmatrix}.$$

Let $(u,v)_\Omega = \int_0^L u \cdot \bar{v}\, dx$, where u may be either scalar or vector valued. Define quadratic forms a and c by

$$\begin{aligned} c(w, \mathbf{v}_{\mathcal{O}}) &= (mw, w)_\Omega + \alpha(w_x, w_x)_\Omega + (\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\mathbf{v}_{\mathcal{O}}, \mathbf{v}_{\mathcal{O}})_\Omega \\ a(w, \mathbf{v}_{\mathcal{O}}) &= K(w_{xx}, w_{xx})_\Omega + (\mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}\mathbf{v}_{\mathcal{O}x}, \mathbf{v}_{\mathcal{O}x})_\Omega + (G_2h_2\varphi, \varphi)_\Omega. \end{aligned}$$

The energy of the beam is given by

$$\mathcal{E}(t) = \frac{R}{2}(c(\dot{w}, \dot{\mathbf{v}}_{\mathcal{O}}) + a(w, \mathbf{v}_{\mathcal{O}}))$$

where R is the width of the beam. The energy "inner product" is defined by

$$< Y, \hat{Y} >_e = a(U; \hat{U}) + c(V; \hat{V}), \qquad (\hat{Y} = (\hat{U}, \hat{V})),$$

where $a(\cdot\,;\cdot)$ and $c(\cdot\,;\cdot)$ are the bilinear forms that coincide with the previously defined quadratic forms $a(\cdot)$, $c(\cdot)$ on the diagonal. Let

$$\begin{aligned} X_1 &= \{u, \mathbf{u}\} \in H^2(0,L) \cap H_0^1(0,L) \times (H^1(0,L))^2 \\ X_0 &= \{u, \mathbf{u}\} \in H_0^1(0,L) \times (L^2(0,L))^2. \end{aligned}$$

It can be shown [2] that the equations of motion are well-posed on the energy space $(U, \dot{U}) \in C([0,T]; X_1 \times X_0)$. It is not hard to prove the same for semigroup solutions. The domain of this semigroup is $X_2 \times X_1$, where

$$X_2 = \{(u, \mathbf{u}) \in X_1 : u \in H^3(0,L),\ \mathbf{u} \in (H^2(0,L))^2 + BC's\}$$

where "+BC's" means $D_x^2 u$ and $D_x \mathbf{u}$ vanish at each end.

Theorem 2. *Let $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$ be as above. Then $\mathcal{A} : \mathcal{D}(\mathcal{A}) \to X_1 \times X_0$ is the generator of a C_0 dissipative semigroup on $X_1 \times X_0$.*

One may formulate the equations of motion (1), (2) as follows:

$$\frac{d}{dt}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 & I \\ A_1 & A_2 \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{B}\{M, \mathbf{g}_{\mathcal{O}}\} \end{pmatrix}; \tag{4}$$

where

$$\mathcal{B}\{M, \mathbf{g}_{\mathcal{O}}\} = \begin{pmatrix} J^{-1} M(t) \delta_L'(x) \\ \mathbf{h}_{\mathcal{O}}^{-1} \mathbf{p}_{\mathcal{O}}^{-1} \mathbf{g}_{\mathcal{O}} \delta_L(x) \end{pmatrix}. \tag{5}$$

In order to define solutions of (4) one first extends the semigroup $e^{\mathcal{A}t}$ to a weaker space defined by duality. However for the inputs defined in (4)-(5), we will later see that (4) is well posed on $X_1 \times X_0$.

3. **Spectral analysis of $\mathcal{A}$.** Setting $\mathcal{A}Y = \lambda Y$ is equivalent to

$$V = \lambda U, \qquad A_1 U + A_2 \lambda U = \lambda^2 U. \tag{6}$$

3.1. **The case $\lambda = 0$.** If $\lambda = 0$ then $V = 0$ and we need to find all solutions to $A_1 U = 0$, or equivalently,

$$\begin{aligned} K D_x^4 u - D_x N h_2 G_2 \varphi &= 0 && (7) \\ -\mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}} D_x^2 \mathbf{u} + \mathbf{B}^T G_2 \varphi &= 0 && (8) \\ \mathbf{B}\mathbf{u} + h_2 N D_x u &= h_2 \varphi. && (9) \end{aligned}$$

Noting that diagonal matrices commute, we multiply (8) from the left by $\mathbf{B}\mathbf{h}_{\mathcal{O}}^{-1}\mathbf{E}_{\mathcal{O}}^{-1}$ and use (9) to obtain

$$-h_2 D_x^2 \varphi + P G_2 \varphi = -h_2 N D_x^3 u; \qquad P = \mathbf{B}(\mathbf{h}_{\mathcal{O}} \mathbf{E}_{\mathcal{O}})^{-1} \mathbf{B}^T > 0. \tag{10}$$

It follows that $(-h_2 D_x^2 + P G_2)$ is a positive, invertible operator on $\mathcal{D}(\mathcal{A})$ and (7) becomes

$$K D_x^4 u - D_x N h_2 G_2 (-h_2 D_x^2 + P G_2)^{-1} (-h_2 N D_x^3 u) = 0. \tag{11}$$

It follows from positivity of the differential operator in (11) (and the boundary conditions for u) that $u = 0$. Hence (10) implies $\varphi = 0$. Thus (9) implies $\mathbf{B}\mathbf{u} = 0$. It follows that $\mathbf{u} = (a + bx)\vec{1}_{\mathcal{O}}$ ($\vec{1}_{\mathcal{O}} = (1,1)^T$) is the general form of $\mathbf{u}$. However, considering the boundary conditions, we conclude the null space is the following

$$\mathbf{u} = \vec{1}_{\mathcal{O}}, \qquad u = 0, \qquad V = 0. \tag{12}$$

Associated with this null vector, one also has a generalized eigenfunction of the form

$$U = 0, \qquad v = 0, \qquad \mathbf{v} = \vec{1}_{\mathcal{O}}. \tag{13}$$

3.2. **Eigenvectors with $u = 0$.** When $u = 0$ we are led to the following system

$$D_x N(G_2\mathbf{B}\mathbf{u} + \tilde{G}_2\mathbf{B}\lambda\mathbf{u}) = 0 \tag{14}$$

$$\lambda^2\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\mathbf{u} - \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}D_x^2\mathbf{u} + \mathbf{B}^T h_2^{-1}(G_2\mathbf{B}\mathbf{u} + \tilde{G}_2\mathbf{B}\lambda\mathbf{u}) = 0. \tag{15}$$

Let $\mathbf{z} = D_x\mathbf{u}$. Applying D_x to (15) leads to

$$\lambda^2\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\mathbf{z} - \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}D_x^2\mathbf{z} = 0.$$

Since $\mathbf{z}$ satisfies $\mathbf{z}(0) = \mathbf{z}(1) = 0$, in order to have nontrivial solutions, we must have that the wave speeds $\sqrt{\frac{E_1}{\rho_1}}$ and $\sqrt{\frac{E_3}{\rho_3}}$ are the same. Since this case is not considered here, there are no solutions of this type. On the other hand, $\mathbf{z}$ could be trivial, in which case $\mathbf{u}$ is constant in each component. Then (14) is satisfied and (15) reduces to

$$\lambda^2\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\mathbf{u} + \mathbf{B}^T h_2^{-1}(G_2\mathbf{B}\mathbf{u} + \tilde{G}_2\mathbf{B}\lambda\mathbf{u}) = 0.$$

This system is equivalent to the following matrix system:

$$\begin{pmatrix} 0 & I \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}\mathbf{B}^T\frac{G_2}{h_2}\mathbf{B} & \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}\mathbf{B}^T\frac{\tilde{G}_2}{h_2}\mathbf{B} \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \lambda\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

The above system, being of order 4, has 4 eigenvalues (up to multiplicity). Two of these correspond to $\lambda = 0$, with $\mathbf{u}$ and $\mathbf{v}$ given by the null vector and generalized null vector as described in (12) and (13). The other two roots satisfy

$$\lambda^2 + \lambda R\tilde{G}_2/h_2 + RG_2/h_2 = 0,\ R = \mathbf{B}\mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}\mathbf{B}^T > 0, \tag{16}$$

where λ is associated with eigenvectors of the form

$$U = (0, h_3\rho_3, -h_1\rho_1)^T, V = \lambda U. \tag{17}$$

In the case that $\tilde{G}_2^2R/h_2 = 4G_2$, $\lambda = -\tilde{G}_2R/(2h_2)$ is a double root and a corresponding generalized eigenvector can be found.

3.3. **All other eigenvectors.** The second equation in (6) can be written

$$m\lambda^2u - \alpha\lambda^2D_x^2u + KD_x^4u - D_xNh_2(G_2 + \lambda\tilde{G}_2)\varphi = 0 \tag{18}$$

$$\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\lambda^2\mathbf{u} - \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}D_x^2\mathbf{u} + \mathbf{B}^T(G_2 + \lambda\tilde{G}_2)\varphi = 0 \tag{19}$$

$$\mathbf{B}\mathbf{u} + h_2ND_xu = \varphi h_2. \tag{20}$$

We look for solutions of the form

$$u = a\sin\sigma_k x, \qquad \mathbf{u} = \vec{C}_k\cos\sigma_k x,; \qquad \sigma_k = k\pi/L, \quad \vec{C}_k = (c_1, c_3)^T.$$

Solutions of this form satisfy all the homogeneous boundary conditions. We seek λ_k and $\vec{C}_k$ so that the system (18)–(20) is satisfied. For simplicity we omit the subscript k. Upon substitution of the modal solutions into (18)–(20) we obtain

$$am\lambda^2 + a\alpha\lambda^2\sigma^2 + aK\sigma^4 + \sigma Nh_2(G_2 + \lambda\tilde{G}_2)(\frac{\mathbf{B}}{h_2}\vec{C} + aN\sigma) = 0 \tag{21}$$

$$\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\lambda^2\vec{C} + \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}\sigma^2\vec{C} + \mathbf{B}^T(G_2 + \lambda\tilde{G}_2)(\frac{\mathbf{B}}{h_2}\vec{C} + aN\sigma) = 0. \tag{22}$$

The system (21)-(22) can be expressed as a standard eigenvalue problem for a six-by-six matrix, and hence there exists for each $\sigma > 0$ six linearly independent eigenvectors and generalized eigenvectors. Define $\Gamma(\lambda) = G_2/\lambda + \tilde{G}_2$. First assume $a = 1$ and solve for $\vec{C}$ in (21)–(22). One obtains the following

$$\vec{C} = -(\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}\lambda^2 + \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}\sigma^2 + \lambda\mathbf{B}^T\Gamma h_2^{-1}\mathbf{B})^{-1}\lambda\mathbf{B}^T\Gamma N\sigma \tag{23}$$

Let $y = \lambda/\sigma$. Upon substitution of (23) into (21) we obtain

$$F(y,\sigma) + G(y,\sigma) = 0, \tag{24}$$

where letting $\tilde{a} = NG_2h_2N, \quad \tilde{b} = N\tilde{G}_2h_2N$ we get

$$\begin{aligned} F &= m\frac{y^2}{\sigma^2} + \alpha y^2 + K + \frac{\tilde{b}y}{\sigma} + \frac{\tilde{a}}{\sigma^2} \\ G &= \frac{N\Gamma\mathbf{B}y^2}{\sigma^2}(\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}y^2 + \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}} + \frac{y}{\sigma}\mathbf{B}^T\Gamma h_2^{-1}\mathbf{B})^{-1}\mathbf{B}^T\Gamma N. \end{aligned} \tag{25}$$

The roots of F are given by:

$$y = (\frac{\tilde{b}}{2\sigma\alpha} + \delta) \pm i\sqrt{\frac{K}{\alpha}} + \epsilon; \qquad \delta = \mathcal{O}(\sigma^{-3}),\ \epsilon = \mathcal{O}(\sigma^{-2}), \tag{26}$$

where δ and ϵ are real. It is also easily shown that if

$$\sqrt{\frac{K}{\alpha}} \notin \{\sqrt{\frac{E_1}{\rho_1}}, \sqrt{\frac{E_3}{\rho_3}}\}, \tag{27}$$

and y is given by (26) then $G(y,\sigma) = \mathcal{O}(\sigma^{-2})$. Hence, using Rouché's Theorem (e.g., as in Hansen[3], or Sect. 4.4 of Luo, Guo, Morgul[7]) one can show that the roots of (24) for k sufficiently large are also given by (26). Hence we obtain that for k sufficiently large, the eigenvalues are given by the following:

$$s_k^{\pm} := \lambda = -\frac{\tilde{b}}{2\alpha} \pm i\sigma_k\sqrt{\frac{K}{\alpha}} + \mathcal{O}(\sigma_k^{-1}). \tag{28}$$

Next we assume C_j, the jth component of $\vec{C}$ is 1 and solve for a in (21)–(22). One finds that

$$a = \frac{\sigma N(G_2 + \tilde{G}_2)\lambda\mathbf{B}\vec{C}}{m\lambda^2 + \alpha^2\sigma^2 + K\sigma^4 + \sigma^2 Nh_2(G_2 + \lambda\tilde{G}_2)N}. \tag{29}$$

Let us set $x = \frac{\lambda}{\sigma}, \quad \Gamma = G_2\lambda + \tilde{G}_2$. Upon substitution of (29) into (21)-(22), we obtain

$$\begin{aligned} &(F(x,\sigma) + \frac{1}{\sigma^2}G(x,\sigma))\vec{C} = 0 \\ &F = \mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}x^2 + \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}} + \frac{1}{\sigma^2}\mathbf{B}^TG_2h_2^{-1}\mathbf{B} + \frac{x}{\sigma}\mathbf{B}^T\tilde{G}_2h_2^{-1}\mathbf{B} \\ &G = \frac{x^2\mathbf{B}^T\Gamma NN\Gamma\mathbf{B}}{(\alpha + \frac{m}{\sigma^2})x^2 + \frac{1}{\sigma}Nh_2\tilde{G}_2Nx + K + \frac{1}{\sigma^2}NG_2h_2N}. \end{aligned} \tag{30}$$

Under the assumption (27), for σ sufficiently large, it is easily checked that G is bounded independent of σ. It can also be easily shown that $(\pm i\sqrt{\frac{E_j}{\rho_j}}, \vec{e}_j)$ (where $\vec{e}_j$, the jth standard basis vector in $\mathbb{R}^2$ and $j = 1, 3$) are solutions of $F(x,\infty)\vec{C} = 0$. Assuming that $\sqrt{\frac{E_1}{\rho_1}} \neq \sqrt{\frac{E_3}{\rho_3}}$, standard perturbation theory of matrices (see for e.g [9]) implies that for σ sufficiently large there exists a solution of (30) near those of $F(x_0,\infty)\vec{e}_j = 0$, where $x_0 = +i\sqrt{\frac{E_j}{\rho_j}}$. Let $x = x_0 + \delta$ and $\vec{V} = \vec{e}_j + \vec{q}$. We seek δ and $\vec{q}$ such that $\vec{q} \perp \vec{e}_j$, $\delta = \mathcal{O}(\sigma^{-1})$, $\vec{q} = \mathcal{O}(\sigma^{-1})$ and $F(x,\sigma)\vec{V} = \mathcal{O}(\sigma^{-2})$. We obtain the following system

$$(\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}(x_0^2 + 2\delta x_o) + \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}} + \frac{(x_0+\delta)}{\sigma}\mathbf{B}^T\tilde{G}_2h_2^{-1}\mathbf{B})(\vec{e}_i + \vec{q}) \quad = \quad \mathcal{O}(\sigma^{-2}). \tag{31}$$

Define $\tilde{\gamma} = \tilde{G}_2/h_2$. An easy calculation shows that $\mathbf{B}^T\tilde{G}_2 h_2^{-1}\mathbf{B}\vec{e}_j := -\tilde{\gamma}\vec{e}_{j^*} + -\tilde{\gamma}\vec{e}_j$, where we use the convention that $1^* = 3, 3^* = 1$. Since e_j is orthogonal to both $\vec{q}$ and $\vec{e}_{j^*}$, from (31) we obtain the following system.

$$\begin{aligned} h_j\rho_j 2\delta x_0 + \frac{x_0+\delta}{\sigma}\tilde{\gamma} &= \mathcal{O}(\sigma^{-2}) \\ \frac{x_0+\delta}{\sigma}(-\tilde{\gamma}\vec{e}_{j^*}) + (\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}}x_0^2 + \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}})\vec{q} &= \mathcal{O}(\sigma^{-2}). \end{aligned}$$

Due to the fact that we have assumed the wave speeds in each layer to be different (27), we may solve for $\vec{q}$. We obtain the following

$$\delta = -\frac{\tilde{\gamma}}{2\sigma h_j\rho_j} + \mathcal{O}(\sigma^{-2}), \vec{q} = \vec{q}(\sigma) = (q_1, q_3)^T, q_j = 0$$

$$q_{j^*} = -\frac{\tilde{\gamma}x_0}{\sigma}(h_{j^*}\rho_{j^*}x_0^2 + h_{j^*}E_{j^*})^{-1} + \mathcal{O}(\sigma^{-2}).$$

Thus we obtain the following eigenvalues for k sufficiently large.

$$\lambda_{k,j}^{\pm} := \lambda = i\sigma_k\sqrt{\frac{E_j}{\rho_j}} - \frac{\tilde{\gamma}}{2h_j\rho_j} + \mathcal{O}(k^{-1}), j = 1, 3. \tag{32}$$

From (29) we also get

$$a = \mathcal{O}(k^{-2}),\ \vec{C} = \vec{e}_j + \vec{q} + \mathcal{O}(k^{-2}). \tag{33}$$

Proposition 1. *Assume the wave speeds are distinct. The spectrum of $\mathcal{A}$ consists of the eigenvalues*

$$\sigma(\mathcal{A}) = \bigcup_{k=0}^{\infty} S_k,$$

where S_0 consists of the double eigenvalue 0 and the roots of (16). Associated eigenvectors and generalized eigenvectors are given in (12)-(13) and (17).

$$S_k = \{\lambda_{k,0}^+ := s_k^+, \lambda_{k,0}^- := s_k^-, \lambda_{k,1}^+, \lambda_{k,1}^-, \lambda_{k,3}^+, \lambda_{k,3}^-\},$$

where for k sufficiently large, $\lambda_{k,j}^+, \lambda_{k,j}^-$ are complex conjugate roots which are given by (28),(32).

For $\lambda = \lambda_{k,j}^{\pm} \neq 0$, the corresponding eigenvectors (and possibly generalized eigenvectors) Y_λ exist and are of the form

$$Y_\lambda = \begin{pmatrix} \frac{U_\lambda}{\lambda} \\ U_\lambda \end{pmatrix};\ U_\lambda = \begin{pmatrix} u_{k,j} \\ \mathbf{u}_{k,j} \end{pmatrix}; \begin{pmatrix} u_{k,j} \\ \mathbf{u}_{k,j} \end{pmatrix} = \begin{pmatrix} A_{k,j}\sin(\sigma_k x) \\ \vec{B}_{k,j}\cos(\sigma_k x) \end{pmatrix},$$

where k sufficiently large the $Y_{\lambda_{k,j}^{\pm}}$ are eigenfunctions given by

$$\begin{pmatrix} u_{k,0} \\ \mathbf{u}_{k,0} \end{pmatrix} = \frac{1}{\sigma_k}\begin{pmatrix} \sin(\sigma_k x) \\ -[(\mathbf{h}_{\mathcal{O}}\mathbf{p}_{\mathcal{O}} + \mathbf{h}_{\mathcal{O}}\mathbf{E}_{\mathcal{O}}y_k^2)^{-1}\mathbf{B}^T\tilde{G}_2 y_k + \mathcal{O}(k^{-1})]\cos(\sigma_k x) \end{pmatrix},$$

where $y_k = \frac{\sigma_k}{s_k^{\pm}}$,

$$\begin{pmatrix} u_{k,1} \\ \mathbf{u}_{k,1} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(k^{-2})\sin(\sigma_k x) \\ \cos(\sigma_k x) \\ \mathcal{O}(k^{-1})\cos(\sigma_k x) \end{pmatrix}, \quad \begin{pmatrix} u_{k,3} \\ \mathbf{u}_{k,3} \end{pmatrix} = \begin{pmatrix} \mathcal{O}(k^{-2})\sin(\sigma_k x) \\ \mathcal{O}(k^{-1})\cos(\sigma_k x) \\ \cos(\sigma_k x) \end{pmatrix}.$$

Remark 2. The fact that the spectrum described in Proposition 1 is exhaustive follows once the Riesz basis property of eigenvectors is proved.

Proof of Theorem 1: Denote by $\tilde{\mathcal{A}}$ the generator $\mathcal{A}$ in the undamped case, i.e., with $\tilde{G}_2 = 0$. For $k \in N$ let $\left(\tilde{\lambda}^{\pm}_{k,0}, \tilde{\lambda}^{\pm}_{k,1}, \tilde{\lambda}^{\pm}_{k,3}\right)$ and $\left(\tilde{Y}^{\pm}_{k,0}, \tilde{Y}^{\pm}_{k,1}, \tilde{Y}^{\pm}_{k,3}\right)$ be the associated eigenvalues and eigenvectors. For $k = 0$ the spectrum of $\tilde{\mathcal{A}}$ consists of the double eigenvalue 0 and the roots of (14)-(15) with with parameter $\tilde{G}_2$ set to 0. The corresponding eigenvectors and generalized eigenvectors are again described by (12)-(13) and (17). It is easy to see that the span of these eigenvectors is the same in the damped and undamped cases. Denote by E_0 the span of these eigenvectors. It is enough to show that the span of the eigenvectors of $\mathcal{A}$ corresponding to $k \in \mathbb{N}$ in Proposition 1 form a Riesz basis for the orthogonal complement of E_0. First note that $\tilde{\mathcal{A}}$ is skew hermitian with respect to the energy inner product $< \cdot, \cdot >_e$ and has compact resolvent. Consequently the eigenvectors of $\tilde{\mathcal{A}}$ are complete and orthogonal with respect to $< \cdot, \cdot >_e$. Furthermore Proposition 1 remains valid for the eigenvectors of $\tilde{\mathcal{A}}$. Therefore, due to the $\mathcal{O}(1)$ difference between $\lambda^{\pm}_{k,j}$ and $\tilde{\lambda}^{\pm}_{k,j}$, it is easy to show that $\sum_{k=1}^{\infty} \sum_{j=1,3} \sum_{+,-} ||\tilde{Y}^{\pm}_{j,k} - Y^{\pm}_{j,k}||^2 < \infty$. Hence by Bari's Theorem, [11] $\{Y_{\lambda^{\pm}_{k,j}}\}$ forms a Riesz basis for the orthogonal complement of E_0 if it can be established that they are ω-independent. Let $E(S)$ denote the span of the eigenvectors and generalized eigenvectors corresponding to $\lambda \in S$. It is easy to see that $E(S_k)$ is orthogonal to $E(S_j)$ for $j \neq k$. (where S_k is defined in Proposition 1). If $\sum_{k=1}^{\infty} \sum_{\lambda \in S_k} c_\lambda Y_\lambda = 0$, then by the block orthogonality we have $\sum_{\lambda \in S_k} c_\lambda Y_\lambda = 0$, $\forall k \in \mathbb{N}$. However for all blocks S_k, the eigenvectors and generalized eigenvectors are known to exist and be linearly independent (as they are derived from a matrix eigenvalue problem). Hence we obtain $c_\lambda = 0$ for all λ. Thus $\{Y_{\lambda^{\pm}_{k,j}}\}$ is an ω-linearly independent set. This completes the proof.

Remark 3. As a consequence of the Riesz basis property, it is easy to show that the semigroup $e^{\mathcal{A}t}$ extends continuously to a C_0 *group* (see e.g Pazy[10]) of operators: $\mathbb{R} \to X_1 \times X_0$.

Proposition 2. *Assume the wave speeds are distinct. Let* $\tilde{G}_2 > 0$ *and*

$$\{\frac{K}{\alpha + \frac{m}{\sigma_k^2}}\}_{k=1}^{\infty} \bigcap \{\frac{E_1}{\rho_1}, \frac{E_3}{\rho_3}\} = \emptyset.$$

Then every non-zero eigenvalue has negative real part.

Proof: If $\{w, \mathbf{v}_{\mathcal{O}}\}$ is a smooth solution of the homogenous problem, then an easy calculation shows that

$$\frac{d\mathcal{E}}{dt} = -h_2 \tilde{G}_2 (\dot{\varphi}, \dot{\varphi})_\Omega$$

and hence for an eigenvalue to exist on the imaginary axis, $\dot{\varphi} = 0$ a.e. Since the eigenvectors satisfy (6), we must have $\lambda\varphi = 0$ in (18),(19), and since $\lambda \neq 0$ we have that $\varphi = 0$. This means eigenvectors satisfy (18),(19) with $G_2 = \tilde{G}_2 = 0$. However, such eigensolutions are readily calculated explicitly. One finds that the only way any of these solutions can have $\varphi = 0$ is if $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$. However this is not possible under the above hypothesis. This completes the proof.

Remark 4. We have assumed the wave speeds to be distinct. However it is interesting to note that if $\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}$ then there is an infinite family of eigenvectors (corresponding to stretching motions) that are completely undamped. Likewise if

the condition given in Proposition 2 fails to be satisfied, it is possible to find appropriate physical parameters so that bending motions exist that do not dissipate energy.

Using the fact that the eigenvectors form a Riesz basis, it is well known (see e.g., Hansen [3]) that the growth of the semigroup $e^{\mathcal{A}t}$ is determined by the real part of the spectrum of $\mathcal{A}$. (This is referred to as the *spectrum-determined growth condition* in the literature; see eg., Chen et. al.[1], Luo Guo, Morgul [7] and references therein for some history and further discusssion.) Thus the following holds.

Corollary 1. *Under the hypothesis of Proposition 2, all finite energy solutions to the homogeneous problem (1)-(2) (with $M(t) = 0$, $\mathbf{g}_{\mathcal{O}}(t) = 0$, with initial data orthogonal to nullvectors (12), (13)) have energy $\mathcal{E}(t)$ that satisfies $\mathcal{E}(t) \leq Me^{-\gamma t}\mathcal{E}(0)$, where M is independent of initial data and $-\gamma < 0$ is the supremum of the real parts of the nonzero eigenvalues of $\mathcal{A}$.*

4. **Exact Controllability Results.** Let us assume that the initial data given is zero, and determine which states are reachable in time T. We write the terminal state as

$$Y(T) = \sum_{\lambda \in \sigma(\mathcal{A})} c_\lambda Y_\lambda; \qquad c_\lambda =< Y(T), Y_\lambda^* >_e$$

where Y_λ^* is the eigenfunction of $\mathcal{A}^*$ with eigenvalue $\bar{\lambda}$. The eigenvectors of A^* are the same as the eigenvectors of $\mathcal{A}$ except for the negative sign corresponding to any term that multiplies $\tilde{G}_2$. We need the following calculation

$$\begin{aligned}
\left\langle \begin{pmatrix} 0 \\ \mathcal{B}\{M, \mathbf{g}_{\mathcal{O}}\} \end{pmatrix}, Y_\lambda^* \right\rangle_e &= c(\mathcal{B}\{M, \mathbf{g}_{\mathcal{O}}\}, V_\lambda^*) = c(\begin{pmatrix} J^{-1}M(t)\delta_L' \\ \mathbf{h}_{\mathcal{O}}^{-1}\mathbf{p}_{\mathcal{O}}^{-1}\mathbf{g}_{\mathcal{O}}\delta_L \end{pmatrix}, \begin{pmatrix} v_\lambda^* \\ \mathbf{v}_\lambda^* \end{pmatrix}) \\
&= < M(t)\delta_L', v_\lambda^* > + < \mathbf{g}_{\mathcal{O}}, \delta_L \mathbf{v}_\lambda^* > \\
&= M(t)b_\lambda + \mathbf{g}_{\mathcal{O}}(t) \cdot \vec{b}_\lambda
\end{aligned}$$

where

$$b_\lambda = \begin{cases} (-1)^k & \text{if } \lambda = s_k^\pm \\ \mathcal{O}(k^{-1}) & \text{if } \lambda = \lambda_{k,j}^\pm, j = 1, 3 \end{cases} \tag{34}$$

$$\vec{b}_\lambda = \begin{cases} \mathcal{O}(k^{-1}) & \text{if } \lambda = s_k^\pm \\ \vec{e}_j g_j(t) + \mathcal{O}(k^{-1}) & \text{if } \lambda = \lambda_{k,j}^\pm, j = 1, 3. \end{cases} \tag{35}$$

Note that all the coefficients in b_λ, $\vec{b}_\lambda$ are bounded. As a consequence of the Carleson measure criterion (see for e.g. Ho and Russell [6] or Hansen and Weiss [5]) (4) is well posed on $X_0 \times X_1$ i.e. given any initial data in $X_0 \times X_1$, and L^2 controls $M(t), \mathbf{g}_{\mathcal{O}}(t)$, there exists a unique solution Y of (4) for which $Y = (U, V)^T \in C([0, T]; X_0 \times X_1)$.

The variation of parameters solution can be written

$$\begin{aligned}
Y(T) &= \int_0^T e^{\mathcal{A}(T-s)} \begin{pmatrix} 0 \\ \mathcal{B}\{M, \mathbf{g}_{\mathcal{O}}\} \end{pmatrix} ds = \int_0^T e^{\mathcal{A}t} \begin{pmatrix} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_{\mathcal{O}}\} \end{pmatrix} ds \\
&\text{where } \tilde{M}(t) = M(T-t), \quad \tilde{\mathbf{g}}_{\mathcal{O}} = \mathbf{g}_{\mathcal{O}}(T-t).
\end{aligned}$$

Hence multiplying the above by the eigenvectors Y_λ^* of $\mathcal{A}^*$ gives

$$\begin{aligned} c_\lambda &= < Y(T), Y_\lambda^* >_e = < \int_0^T e^{\mathcal{A}t} \begin{pmatrix} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_{\mathcal{O}}\} \end{pmatrix}, Y_\lambda^* \, ds >_e \\ &= \int_0^T e^{\lambda t} < \begin{pmatrix} 0 \\ \mathcal{B}\{\tilde{M}, \tilde{\mathbf{g}}_{\mathcal{O}}\} \end{pmatrix}, Y_\lambda^* >_e ds \\ &= \begin{cases} \int_0^T e^{\lambda t}(\tilde{M}(t)(-1)^k + \tilde{\mathbf{g}}_{\mathcal{O}} \cdot \vec{\epsilon}_k)\, ds & \text{if } \lambda = s_k^\pm \\ \int_0^T e^{\lambda t}(\tilde{M}(t)\delta_{k,j} + (\tilde{\mathbf{g}}_{\mathcal{O}})_j + \tilde{\epsilon}_{k,j}(\tilde{\mathbf{g}}_{\mathcal{O}})_{j^*} ds & \text{if } \lambda = \lambda_{kj}^\pm, j = 1, 3, \end{cases} \end{aligned}$$

where all the terms $\delta_{k,j}$, $\vec{\epsilon}_k$, $\tilde{\epsilon}_{kn}$ are $\mathcal{O}(k^{-1})$. For $k \in \mathbb{N}$, this results in the three moment problems

$$c_{0,k}^\pm = \int_0^T e^{\lambda_{0,k} t}(\tilde{M}(t) + \tilde{\mathbf{g}}_{\mathcal{O}} \cdot \vec{\epsilon}_k)\, ds \tag{36}$$

$$c_{1,k}^\pm = \int_0^T e^{\lambda_{1,k} t}(\delta_{1,k}\tilde{M}(t) + (\tilde{\mathbf{g}}_{\mathcal{O}})_1 + \tilde{\epsilon}_{k,1}(\tilde{\mathbf{g}}_{\mathcal{O}})_3, ds \tag{37}$$

$$c_{3,k}^\pm = \int_0^T e^{\lambda_{3,k} t}(\delta_{3,k}\tilde{M}(t) + (\tilde{\mathbf{g}}_{\mathcal{O}})_3 + \tilde{\epsilon}_{k,3}(\tilde{\mathbf{g}}_{\mathcal{O}})_1 ds, \tag{38}$$

and a finite system corresponding (respectively) to the nullvector (12), the generalized nullvector (13), and the eigenvectors described in (17):

$$c_{0,1} = \int_0^T 0\, ds \tag{39}$$

$$c_{0,2} = \int_0^T \tilde{g}_1 + \tilde{g}_3\, ds \qquad c_{0,3}^\pm = \int_0^T e^{\lambda_{0,3}^\pm}(h_3\rho_3\tilde{g}_1 - h_1\rho_3 g_3)\, ds \tag{40}$$

Remark 5. As is easy to see from (39), it is not possible to steer a solution of (1), (2) from the origin to the state corresponding to the null vector solution $w = 0$, $\mathbf{v}_{\mathcal{O}} = (1, 1)^T$.

Theorem 3. *Assume the wave speeds are distinct. Given any $\{c_\lambda\} \in \ell^2$ there exists functions $M(t)$ and $\mathbf{g}_{\mathcal{O}}(t)$ in $L^2(0, T)$ which solve the three moment problems (36)–(38), for all $k > K$, where K is sufficiently large in any time $T > \tau$, where the control time τ is given in (3).*

Proof: The proof is the same as the proof for the case when only (36) and (37) are present. Thus to give the idea we consider the following isomorphic coupled moment problem:

$$a_k = \int_0^T e^{s_k t}(f(t) + \delta_k g(t))\, dt, \quad b_k = \int_0^T e^{\lambda_k t}(\epsilon_k f(t) + g(t))\, dt \quad k \in \mathbb{Z}. \tag{41}$$

It is assumed that the sequences $\{s_k\}$ and $\{\lambda_k\}$ are each (individually) sequences of complex numbers lying in a vertical strip of $\mathbb{C}$ such that

$$\lim_{|k|\to\infty} (\operatorname{Im} s_k) - Ak = 0, \qquad \lim_{|k|\to\infty} (\operatorname{Im} \lambda_k) - Bk = 0; \; A > 0, \, B > 0.$$

Under these conditions, there is a uniform separation of the eigenvalues in each sequence for $|k| > K_0$ sufficiently large. Hence it is well-known (e.g., Young[11]) that $\{e^{s_k t}\}_{|k|\geq K_0}$ forms a Riesz basis for its closure in $L^2(0, T)$, for any $T > 2\pi/A$. Similarly $\{e^{\lambda_k t}\}_{k\geq K}$ forms a Riesz basis for its closure in $L^2(0, T)$, for any $T >$

$2\pi/B$. For the moment, let us assume the separation condition holds for all k. We first prove that if the sequences $\{\delta_k\}$ and $\{\epsilon_k\}$ have sufficiently small ℓ^2 norms, then there exists a solution to (41). Let $T > 2\pi/A$, $T > 2\pi/B$. Let $\{p_k\}$, $\{q_k\}$ be the biorthogonal sequences to $\{e^{s_k t}\}$, $\{e^{\lambda_k t}\}$. We seek f, g which solve

$$\begin{pmatrix} f \\ g \end{pmatrix} = K \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} \sum_{k=1}^{\infty} [a_k - \delta_k < e^{s_k t}, g >] p_k(t)\, dt \\ \sum_{k=1}^{\infty} [b_k - \epsilon_k < e^{\lambda_k t}, f >] q_k(t)\, dt \end{pmatrix}. \tag{42}$$

Since $\{\delta_k\}$ and $\{\epsilon_k\}$ are ℓ^2 sequences, and $\{e^{s_k t}\}$, $\{e^{\lambda_k t}\}$ form an $L - basis$ in $L^2(0,T)$, if follows that K maps $L^2(0,T)$ into itself. We show K is a contraction:

$$\begin{aligned}
\|K \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} &- K \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \|^2_{L^2(0,T)} \\
&= \| \sum_{k=1}^{\infty} \begin{pmatrix} -\delta_k < e^{s_k t}, g_1 - g_2 > p_k \\ -\epsilon_k < e^{\lambda_k t}, f_1 - f_2 > q_k(t) \end{pmatrix} \|^2_{L^2(0,T)} \\
&\leq C(\|\{\delta_k < e^{s_k t}, g_2 - g_1 >\}\|^2_{\ell^2} + \|\{\epsilon_k < e^{\lambda_k t}, f_2 - f_1 >\}\|^2_{\ell^2}) \\
&\leq C(\sum_{k=1}^{\infty} |\delta_k|^2 | < e^{s_k t}, g_2 - g_1 > |^2 + |\epsilon_k|^2 | < e^{\lambda_k t}, f_2 - f_1 > |^2 \\
&\leq CT \sum_{k=1}^{\infty} |\delta_k|^2 \|g_2 - g_1\|^2 + |\epsilon_k|^2 \|f_2 - f_1\|^2 \\
&\leq CT(\sum_{k=1}^{\infty} |\delta_k|^2 + |\epsilon_k|^2) \| \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} - \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \|^2_{L^2(0,T)}.
\end{aligned}$$

Thus if the ℓ^2 norms of ϵ_k and δ_k are small enough, K is a contraction. Since $\{\delta_k\}$ and $\{\epsilon_k\}$ are $\mathcal{O}(k^{-1})$, it follows that for N sufficiently large, we can find $f = f_N$ and $g = g_N$, both in $L^2(0,T)$, which solves (41) for $|k| \geq N$. Applying the same idea to (36)–(38) proves the theorem. The control time (3) is obtained by $\tau = \max\,(2\pi/A, 2\pi/B, 2\pi/C)$, where $A = \frac{\pi}{L}\sqrt{\frac{K}{\alpha}}$, $B = \frac{\pi}{L}\sqrt{\frac{E_1}{\rho_1}}$, $C = \frac{\pi}{L}\sqrt{\frac{E_3}{\rho_3}}$.

Keeping Remark 5 in mind, one can ask whether it is possible to solve (36)–(38) for all k together with (40). If possible, then exact controllability of (1)-(2) holds modulo the uncontrollable quotient described in Remark 5. A sufficient condition for this result is that the eigenvalues grow (are not repeated) along each branch. It is not hard to show that this will hold if the coupling between the equations is sufficiently small. Thus one can prove the following.

Corollary 2. *If G_2 and $\tilde{G}_2$ are sufficiently small, then for $T > \tau$ there exists $M(t)$ and $\mathbf{g}_{\mathcal{O}}(t)$ in $L^2(0,T)$ which solve the moment problems (36)–(38), (40).*

As a consequence of Theorem 3, Corollary 2 and regularity (admissibility) results described earlier, we can summarize the main controllability results as follows.

Theorem 4. *Assume the wave speeds are distinct. Given any initial data $Y_0 = (U_0, V_0) \in X_1 \times X_0$ and $T > \tau$ there exist square-integrable controls $M(t)$, $\mathbf{g}_{\mathcal{O}}(t)$ such that the solution $Y(t)$ to (4) satisfies*

$$Y = (U, V) \in C([0,\infty); X_1 \times X_0)$$

and there exists $R_0 \in \mathbb{N}$ such that for $t \geq T$

$$Y \in \mathcal{M} := span\ \{Y_\lambda : \lambda \in \sigma(\mathcal{A}), |\lambda| \leq R_0\}.$$

Furthermore, if G_2 and $\tilde{G}_2$ are sufficiently small, then $\mathcal{M}$ reduces to the one-dimensional space generated by the inertial solution $w \equiv 0$, $v_1 = v_3 =$ constant.

Remark 6. Equivalently, given zero initial data ($Y_0 = 0$), and for any $T > \tau$ terminal target data Y_T, there exist controls $M(t)$, $\mathbf{g}_{\mathcal{O}}(t)$ such that the solution at time T satisfies $Y(T) - Y_T \in \mathcal{M}$. The equivalence of this form of reachability and the null-controllability in Theorem 4 follows from the invertibility of the semigroup (see Remark 3).

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