**Project**

**Hw 5 prob. 3.** Let $A$ be the operator defined by $Au = -[(1 + |x|)u'(x)]'$ on the domain of functions in $H^2(\Omega)$ that are zero at the end points. Show that $A$ is coercive and self-adjoint.

(Extension of problem 3. Explain why $L_A$ must be an eigenvalue of $A$ (To prove this you need to consider the inverse operator and apply a result from Ch.6). Also try to find a good upper bound for the first eigenvalue of $A$ by plugging some functions belonging to $H(\Omega)$.)

**Self-Adjoint:** Compute $\langle Au, v \rangle = \langle u, A^*v \rangle$, where $Au = -[(1 + |x|)u'(x)]'$ and $D_A = \{ H^2(a, b) \mid u(a) = u(b) = 0 \}$.

\[
\langle Au, v \rangle = \langle -[(1 + |x|)u'(x)]', v \rangle = \int_a^b -[(1 + |x|)u'(x)]'v(x)dx
\]

int by parts  $= -v(x)(1 + |x|)u'(x)|_a^b + \int_a^b (1 + |x|)u'(x)v'(x)dx$

int by parts  $= -v(x)(1 + |x|)u'(x)|_a^b + u(x)(1 + |x|)v'(x)|_a^b + \int_a^b [(1 + |x|)v'(x)]'u(x)dx$

We require that $v \in H^2(a, b)$ with zero boundary conditions and since $u(a) = u(b) = 0$

\[
\underbrace{-v(x)(1 + |x|)u'(x)|_a^b + u(x)(1 + |x|)v'(x)|_a^b}_{= 0} + \underbrace{\int_a^b [(1 + |x|)v'(x)]'u(x)dx}_{= 0} = \int_a^b [(1 + |x|)v'(x)]'u(x)dx = \langle u, A^*v \rangle
\]

This gives that $A^* = -[(1 + |x|)v'(x)]'$ with $D_{A^*} = \{ H^2(a, b) \mid v(a) = v(b) = 0 \}$. Hence $A$ is self-adjoint \(\square\)

**Coercive:** The symmetric operator $A$ is coercive if $\langle Au, v \rangle \geq c||v||^2$, $c > 0$ for all $v \in D_A$

\[
\langle Av, v \rangle = \langle -[(1 + |x|)v'(x)]', v \rangle = \int_a^b -[(1 + |x|)v'(x)]'v(x)dx
\]

int by parts  $= -v(x)(1 + |x|)v'(x)|_a^b + \int_a^b (1 + |x|)(v'(x))^2dx = \int_a^b (1 + |x|)(v'(x))^2dx$

$\geq \int_a^b (v'(x))^2dx = \int_a^b |v'(x)|^2dx = ||v'(x)||^2$

We then obtain the following inequality,

\[
\langle Av, v \rangle \geq ||v'(x)||^2
\]  \(\text{ (1)}\)

Now consider

\[
|v(x)|^2 = \left| \int_a^x v'(x)dx \right|^2 \leq \left( \int_a^x 1^2dt \right) \left( \int_a^x |v'(t)|^2dt \right) \leq (x-a) \int_a^x |v'(t)|^2dt \leq (x-a) \int_a^b |v'(t)|^2dt = (x-a)||v'(x)||^2
\]

This gives

\[
|v(x)|^2 \leq (x-a)||v'(x)||^2
\]

Now integrating both sides w.r.t $x$ from $a$ to $b$ gives us the following
\[
\int_a^b |v(x)|^2 \, dx \leq \int_a^b (x-a)||v'(x)||^2 \, dx = ||v'(x)||^2 \int_a^b (x-a) \, dx = \frac{(b-a)^2}{2} ||v'(x)||^2
\]

The above inequality gives us that
\[
||v(x)||^2 \leq \frac{(b-a)^2}{2} ||v'(x)||^2 \iff \quad ||v'(x)||^2 \geq \frac{2}{(b-a)^2} ||v(x)||^2
\]

Hence we have the following inequality
\[
||v'(x)||^2 \geq c ||v(x)||^2, \quad c = \frac{2}{(b-a)^2}
\]

By using (1) and (2)
\[
\langle Av, v \rangle \geq ||v(x)||^2 \geq c ||v(x)||^2 \iff \quad \langle Av, v \rangle \geq \frac{2}{(b-a)^2} ||v(x)||^2
\]

This gives that \( \langle Av, v \rangle \geq c \) for all \( v \in D_A \). Hence \( A \) is coercive \( \square \)

**Inverse of A:** Consider the problem \( Au = f(x) \) with \( u(a) = u(b) = 0 \) and following the Green’s Formula problem
\[
-(1 + |x|)g'(x, \xi) = \delta(x - \xi), 
\quad g(a, \xi) = g(b, \xi) = 0, 
\quad a \leq x \leq b
\]

The solution to (3) is given by
\[
g(x, \xi) = \begin{cases} 
\int_a^x \frac{c_1}{1+|s|} \, ds, & a \leq x < \xi \leq b \\
\int_a^x \frac{c_1}{1+|s|} \, ds + \int_\xi^x \frac{1}{1+|s|} \, ds, & a \leq \xi < x \leq b
\end{cases}
\]

For the case \( \xi < 0 \)
\[
c_1 = \frac{\ln|1 - \xi|}{\ln|1 + b|} - \frac{\ln|1 - \xi|}{\ln|1 - a|}
\]

For the case \( \xi \geq 0 \)
\[
c_1 = \frac{\ln|1+|b|}{\ln|1+\xi|} - \frac{\ln|1+|b||}{\ln|1-a|}
\]

The solution to the problem \( Au(x) = f(x) \) is then given by \( u(x) = \int_a^b g(x, \xi) f(\xi) \, d\xi \). Hence we can define
\[
Kf = \int_a^b g(x, \xi) f(\xi) \, d\xi \iff K = A^{-1}
\]

\( K \) is an integral operator with kernel \( g(x, \xi) \), which is continuous and therefore bounded. Kernel is Hilbert-Schmidt by definition we have that \( K \) is a Hilbert-Schmidt operator and by Theorem on p.357(Stakgold) \( K \) is a compact operator.

\( K \) is compact, self-adjoint, symmetric and positive. The results we have from ch.6 relate to the inverse properties between \( A \) and \( K \) in the eigenvalue sense. Since \( K \) is compact we have that \( \lambda_1, \lambda_2, \ldots \to 0 \) as \( n \to \infty \). \( \lambda_1 \) is positive and its the largest eigenvalue of \( K \). This provides the information on smallest eigenvalue of \( A \). That is \( \lambda_1 = \frac{1}{\Lambda_1} \) where \( \Lambda_1 \) is the smallest eigenvalue of \( A \). For the operator \( A \) the eigenvalues are \( \Lambda_1, \Lambda_2, \ldots \to \infty \), as \( n \to \infty \).

Going back to the question of why \( L_A \) must be an eigenvalue of \( A \)
\[
L_A \equiv \inf_{v \in D_A} R(v) = \inf_{v \in D_A} \langle Av, v \rangle
\]

Now we will consider two theorems (5.8.5) and (5.8.8) both in Stakgold.
**Thm 5.8.5** Let $A$ be symmetric and bounded below. If there is an element $u$ in $D_A$ for which the infimum in (4) is attained, $(L_A, u)$ is an eigenpair and $L_A$ is the lowest eigenvalue of $A$.

**Thm 5.8.8** If K is a symmetric compact operator, then if $U_K \neq 0$, it is an eigenvalue, and if $L_K \neq 0$, it is an eigenvalue.

On the previous page we talked about the compactness of $K$ and we talked about the eigenvalues of $K$ which go to zero as $n$ goes to infinity. We can talk about the largest eigenvalue of $K$. $\lambda_1$ is the largest eigenvalue of $K$ and is NOT zero. By thm 5.8.8 $U_k \neq 0$. Moreover, $U_k$ is attained which gives us that $L_A$ is attained. Since $L_A$ is attained then by thm 5.8.5 $L_A$ must be an eigenvalue of $A$. □

**eigenvalue approximation** Now we would like to find a good upper bound for the first eigenvalue of $A$ and we will attempt this by making use of the Rayleigh quotient and some suitable function on the domain of $A$. For the following computations we will consider the interval $[-1, 1]$.

Rayleigh quotient is defined by

$$R(v) = \frac{\langle Av, v \rangle}{\|v\|^2}, \quad v \in D_A, v \neq 0$$

The functions used are shown on the table below. The computations were obtained by writing a script that looped over a list of functions and each time the iteration ran the Rayleigh quotient of each function was computed. I used Matlab and Mathematica.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$u_n(x)$</th>
<th>$R(u)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$-(x + 1)(x - 1)$</td>
<td>4.3750</td>
</tr>
<tr>
<td>2</td>
<td>$-(x + 1)^2(x - 1)$</td>
<td>5.9609</td>
</tr>
<tr>
<td>3</td>
<td>$-(x + 1)(x - 1)^2$</td>
<td>5.9609</td>
</tr>
<tr>
<td>4</td>
<td>$-(x + 1)^4(x - 1)^4$</td>
<td>6.7648</td>
</tr>
<tr>
<td>5</td>
<td>$\cos \frac{\pi}{2}x$</td>
<td>4.2011</td>
</tr>
<tr>
<td>6</td>
<td>$-(x + 1)(x - 1)\cos \frac{\pi}{2}x$</td>
<td>4.7766</td>
</tr>
<tr>
<td>7</td>
<td>$-(x + 1)(x - 1)\cos \frac{\pi}{2}x$</td>
<td>4.1438</td>
</tr>
<tr>
<td>8</td>
<td>$-(x + 1)(x - 1)\cos x$</td>
<td>4.1381</td>
</tr>
<tr>
<td>9</td>
<td>$-(x + 1)(x - 1)(\cos \frac{\pi}{2})^4$</td>
<td>4.1329</td>
</tr>
<tr>
<td>10</td>
<td>$-(x + 1)(x - 1)(x - 10)^2(x + 10)^2(\cos \frac{\pi}{2})^4$</td>
<td>4.1323</td>
</tr>
</tbody>
</table>

**eigenvalue upper bound** From the table above we can choose the function $u(x)$ that has the smallest $R(u)$ value. In this case we have $\lambda = 4.1323$. This eigenvalue might provide a good upper bound for the smallest eigenvalue of $A$.

**problem extension?** Let’s consider the 4 best functions from the table and try obtain the best possible linear combination of such functions to see if we can get a lower eigenvalue and hence a better upper bound for $A$. 
Now we verify that we have obtained a better upper bound for the first eigenvalue of A. We form
\[ \langle Au, w_j \rangle = \Lambda \langle u, w_j \rangle, \quad j = 1, 2, 3, \ldots, k \]
and setting \( u(x) = \sum_{i=1}^{k} c_i w_i \), we obtain
\[ \sum_{i=1}^{k} \langle Aw_i, w_j \rangle c_i = \Lambda \sum_{i=1}^{k} \langle w_i, w_j \rangle c_i \quad j = 1, 2, 3, \ldots, k \]

For the numerical example we will consider the best 4 functions so \( k=4 \). Here \( u_7(x), u_8(x), u_9(x) \) and \( u_{10}(x) \) will be the functions corresponding to \( w_1(x), w_2(x), w_3(x) \) and \( w_4(x) \). We need to set up the appropriate eigenvalue problem and solve for the eigenvalues and the eigenvectors.

The eigenvalue problem has this form
\[ A\vec{c} = \Lambda B \vec{c} \]

The matrices can be illustrated as follows
\[
\begin{pmatrix}
\langle Aw_1, w_1 \rangle & \langle Aw_2, w_1 \rangle & \langle Aw_3, w_1 \rangle & \langle Aw_4, w_1 \rangle \\
\langle Aw_1, w_2 \rangle & \langle Aw_2, w_2 \rangle & \langle Aw_3, w_2 \rangle & \langle Aw_4, w_2 \rangle \\
\langle Aw_1, w_3 \rangle & \langle Aw_2, w_3 \rangle & \langle Aw_3, w_3 \rangle & \langle Aw_4, w_3 \rangle \\
\langle Aw_1, w_4 \rangle & \langle Aw_2, w_4 \rangle & \langle Aw_3, w_4 \rangle & \langle Aw_4, w_4 \rangle \\
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{pmatrix}
= \Lambda
\begin{pmatrix}
\langle w_1, w_1 \rangle & \langle w_2, w_1 \rangle & \langle w_3, w_1 \rangle & \langle w_4, w_1 \rangle \\
\langle w_1, w_2 \rangle & \langle w_2, w_2 \rangle & \langle w_3, w_2 \rangle & \langle w_4, w_2 \rangle \\
\langle w_1, w_3 \rangle & \langle w_2, w_3 \rangle & \langle w_3, w_3 \rangle & \langle w_4, w_3 \rangle \\
\langle w_1, w_4 \rangle & \langle w_2, w_4 \rangle & \langle w_3, w_4 \rangle & \langle w_4, w_4 \rangle \\
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{pmatrix}
\]

Note: If \( w_1, w_2, w_3, w_4 \) are orthonormal then computations are reduced to
\[
\begin{pmatrix}
\langle Aw_1, w_1 \rangle & \langle Aw_2, w_1 \rangle & \langle Aw_3, w_1 \rangle & \langle Aw_4, w_1 \rangle \\
\langle Aw_1, w_2 \rangle & \langle Aw_2, w_2 \rangle & \langle Aw_3, w_2 \rangle & \langle Aw_4, w_2 \rangle \\
\langle Aw_1, w_3 \rangle & \langle Aw_2, w_3 \rangle & \langle Aw_3, w_3 \rangle & \langle Aw_4, w_3 \rangle \\
\langle Aw_1, w_4 \rangle & \langle Aw_2, w_4 \rangle & \langle Aw_3, w_4 \rangle & \langle Aw_4, w_4 \rangle \\
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{pmatrix}
= \Lambda
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{pmatrix}
\]

The above eigenvalue problem is very similar to the one we do in Homework 5. For this project \( w_1, w_2, w_3, w_4 \) are not orthonormal. In any case we proceed to find the eigenvalues with the corresponding eigenvectors. We again use numerical tools found in Mathematica and Matlab to find the eigenvalues. We find the minimum eigenvalue
\[ \Lambda_1 = 4.1262 \]

The corresponding eigenvector
\[
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\end{pmatrix}
= \begin{pmatrix}
0.3652 \\
0.8980 \\
-0.2454 \\
-0.00004645 \\
\end{pmatrix}
\]

Now we verify that we have obtained a better upper bound for the first eigenvalue of A. We form \( u(x) \)
\[
u(x) = \sum_{i=1}^{4} c_i w_i = c_1 w_1 + c_2 w_2 + c_3 w_3 + c_4 w_4
\]
\[
= (1 - x^2) \left( -0.7099 + 0.00929 x^2 - 0.00004645 x^4 \right) \cos \left( \frac{x}{2} \right)^4 + 0.89 \cos x - 0.3652 \cos \frac{\pi x}{3}
\]

We finally use the above \( u(x) \) and use it on the Rayleigh quotient (5) and we obtain that \( R(u(x)) = 4.126097 \) Comparing this result and the results from Table 1 we observe that this value is smaller than all those results computed before. We were able to get a linear combination of the best four functions and we achieved a good upper bound for the first eigenvalue of A. Hence an upper bound for first eigenvalue of A is 4.126097 \( \square \)