Exercise 6.5.5: Show that the BVP

\[-u'' + 4\pi^2 \int_0^1 u(x)dx = \lambda u, \quad 0 < x < 1; \quad u(0) = u(1), \quad u'(0) = u'(1) \quad (5.40)\]

has \( \lambda = 4\pi^2 \) as an eigenvalue of multiplicity 3.

Proof. First we show that the operator \( Au = -u'' + 4\pi^2 \int_0^1 u(x)dx \) is symmetric and positive.

Let \( u, v \in D_A \). By applying boundary conditions on \( u \) and \( v \):

\[< Au, v > = \int_0^1 Auvdx = \int_0^1 -u''vdx + 4\pi^2 \int_0^1 (\int_0^1 u(y)dy)v(x)dx \]

\[= -uv'|_0^1 + \int_0^1 v'u'dx + 4\pi^2 \int_0^1 u(x)(\int_0^1 v(y)dy)dx \]

\[= v'u'|_0^1 - \int_0^1 uv''dx + 4\pi^2 \int_0^1 u(x)(\int_0^1 v(y)dy)dx \]

\[=< u, Av > \]

Now we show that \( A \) is positive, rather coercive. Let \( u \in D_A \). Following p 342 of textbook, we know

\[\int_0^1 |u'(x)|^2dx \geq 2||u||^2\]

Hence

\[< Au, u > = \int_0^1 -u''udx + 4\pi^2 \int_0^1 (\int_0^1 udy)udx \]

\[= -uu'|_0^1 + \int_0^1 |u'|^2dx + 4\pi^2(\int_0^1 udx)^2 \geq \int_0^1 |u'(x)|^2dx \geq 2||u||^2\]

Hence \( A \) is symmetric and coercive for \( ||u|| \neq 0 \), thus, all its eigenvalues are positive.

Now the BVP is:

\[-u'' + 4\pi^2 \alpha = \lambda u, \quad \alpha = < 1, u > \]

\[u'' + \lambda u = 0\]
\[ u = c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x \]

Applying the boundary conditions on \( u \) we get

\[
\begin{pmatrix}
\sin \sqrt{\lambda} & \cos \sqrt{\lambda} - 1 \\
1 - \cos \sqrt{\lambda} & \sin \sqrt{\lambda}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
\det \begin{pmatrix}
\sin \sqrt{\lambda} & \cos \sqrt{\lambda} - 1 \\
1 - \cos \sqrt{\lambda} & \sin \sqrt{\lambda}
\end{pmatrix} = 0
\]

\[
\Rightarrow \sin^2 \sqrt{\lambda} + (1 - \cos \sqrt{\lambda})^2 = 0
\]

\[
\Rightarrow \cos \sqrt{\lambda} = 1 = \cos 2n\pi
\]

\[
\Rightarrow \lambda = 4n^2\pi^2 \quad n = 1, 2, 3, ...
\]

Now WLOG, let \( \alpha = \langle 1, u \rangle = 1 \)

Hence,

\[-u'' + 4\pi^2 = \lambda u\]

\[
\Rightarrow -u'' = \lambda u - 4\pi^2
\]

\( u_P = A, (let), \quad u'_P = 0 \)

\[
\Rightarrow u_P = \frac{4\pi^2}{\lambda}
\]

\[
\Rightarrow u = c_1 \sin 2\pi nx + c_2 \cos 2\pi nx + \frac{4\pi^2}{\lambda}
\]

Hence the eigenfunctions are \( 1, \sin 2\pi nx, \cos 2\pi nx \). corresponding to the eigenvalue \( 4\pi^2n^2 \), where \( n = 1, 2, 3, ... \)

Hence both the cases for \( \alpha = 0 \) and \( \alpha = 1 \) are consistent.

Now we look into the \( \lambda = 0 \) case.

\[-u'' = 0\]

\[
\Rightarrow u = a + bx + cx^2
\]

Applying boundary conditions on \( u \) we get

\( a = b = c = 0 \)

Hence \( \lambda = 0 \notin \sigma_P(A) \).

Hence it is proved that \( \lambda = 4\pi^2 \) is an eigenvalue of \( A \) when \( n = 1 \) and its multiplicity is 3.
Exercise 6.5.6: Find all the eigenvalues and eigenfunctions of (5.40). Does the set of eigenfunctions form a basis?

Solution: From the previous problem we have seen that the eigenvalues of the operator A are $4\pi^2n^2$, where $n = 1, 2, 3, \ldots$ and the corresponding eigenfunctions are $1, \cos 2\pi nx$ and $\sin 2\pi nx$.

For the eigenfunctions to form a basis for the operator, we need to check if we can change the integro-differential operator A into an integral operator K and see if K is compact or not.

We try to find the Greens function for the BVP:

$$-w''(x) = h(x), \quad w(0) = w(1), \quad w'(0) = w'(1)$$

The Greens function equation for this problem is:

$$-g''(x, \xi) = \delta(x - \xi) - 1, \quad g(0, \xi) = g(1, \xi), \quad g'(0, \xi) = g'(1, \xi), \quad 0 < x, \xi < 1$$

The solution is given by the modified Greens function

$$g_M(x, \xi) = \frac{1}{12} + \frac{(x-\xi)^2}{2} - \frac{1}{3}||x - \xi||, \quad 0 < x, \xi < 1.$$ 

Our next step is to find the Greens function for the BVP:

$$-u''(x) + 4\pi^2 \int_0^1 u dx = f(x) \quad (1)$$

We write $f = f_{avg} + g$, where $f_{avg} = \int_0^1 f dx$, and $g = f - f_{avg}$.

Hence, $\int_0^1 g dx = 0$.

Let $u(x) = u_1(x) + u_2(x)$, where

$$-u_1''(x) + 4\pi^2 \int_0^1 u_1 dx = f_{avg}(x) \quad (2)$$

and

$$-u_2''(x) + 4\pi^2 \int_0^1 u_2 dx = g(x) \quad (3)$$

(2) $\implies -u_1'' + 4\pi^2 a = f_{avg}$, where $a = \int_0^1 u_1 dx$

$\implies -u_1'' = c = f_{avg} - 4\pi^2 a$

$$u_1 = -\frac{cx^2}{2} + \alpha x + \beta$$

$u_1(0) = \beta = u_1(1) = -\frac{c}{2} + \alpha + \beta$

$\implies \alpha = \frac{c}{2}$
\[ u'_1(0) = \alpha = u'_1(1) = -c + \alpha \]
\[ \implies c = 0 \implies f_{avg} - 4\pi^2 a = 0 \implies a = \frac{f_{avg}}{4\pi^2} \]
\[ \implies \int_0^1 u_1 dx = \frac{f_{avg}}{4\pi^2} \]

Now \( u_1 = \beta \implies \int_0^1 \beta dx = \beta = \frac{f_{avg}}{4\pi^2} \)

Hence \( u_1 = \frac{f_{avg}}{4\pi^2} \)

For \( u_2 \) we use the Greens function \( g_M(x, \xi) \).

\[ u_2(x) = G_M(g - 4\pi^2 \int_0^1 u_2 dx) + \gamma \]

Since \( \int_0^1 g dx = 0 \implies \int_0^1 u_2 dx = 0 \)

The value of \( \gamma \) that satisfies \( \int_0^1 u_2 dx = 0 \) is \( \gamma = 0 \).

Now this choice does not wreck the previous decomposition.

Hence, \( u_2(x) = G_M(g - 4\pi^2 \int_0^1 u_2 dx) \)

Since, \( G_M 1 = 0 \)

Thus \( u_2 = G_M g \)

Moreover this satisfies the consistency
\[ \int_0^1 u_2 dx = \int_0^1 G_M g dx = \int_0^1 (\int_0^1 g_M(x, \xi) .1 dx) g(\xi) d\xi = 0 \]

Hence \( u_2 \) satisfies
\[ -u_2 + 4\pi^2 \int_0^1 u_2 dx = g \]

Hence
\[ u(x) = u_1(x) + u_2(x) = G_M g + \frac{f_{avg}}{4\pi^2} \]

Let \( \tilde{g}(x, \xi) = g_M(x, \xi) - \frac{1}{4\pi^2} f(x) d\xi \)

Let \( \tilde{g}(x, \xi) = g_M(x, \xi) - \frac{1}{4\pi^2} \)
\[ u(x) = \int_{0}^{1} \tilde{g}(x, \xi)f(\xi)d\xi \]

Clearly, \( \tilde{g} \) is Hilbert-Schmidt and hence the integral operator \( Kf = u \) where

\[ Kf = \int_{0}^{1} \tilde{g}(x, \xi)f(\xi)d\xi \]

is compact. Since \( A \) is symmetric and coercive, so is \( K \).

Thus the eigenfunctions form a basis.