Practice problems: Week 1  math519

1. Solve the Initial Value Problem \( \frac{dx}{dt} = tx^3, \quad x(0) = a \) and give the interval on which the solution exists.

2. Let \( a = 1 \) and convert the above IVP into a Volterra integral equation. This is an equation of the form \( x = T(x) \). One can sometimes use iteration: \( x_{n+1} = T(x_n) \) to approximate the solution. Try this with zero for \( x_0 \), and compute up to \( x_2 \).

3. For \( \frac{dx}{dt} = (\sin(xt))^{1/3}, \quad x(a) = b \), determine all values of \( a, b \) for which there exists a local solution and also all values for which a unique local solution is known to exist. (Use the existence-uniqueness theorem.)

4. Find the general solution:

\[
y'''' - 9y'' + 27y' - 27y = 0
\]

5. Show that the Laplacian operator in two dimensions is rotationally invariant, i.e., let

\[
\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

and change variables from \((x, y)\) to \((s, t)\) (using chain rule) and show that the Laplacian operator is unchanged.

6. Find a linear change of variables like in prob. 5 that reduces the one-dimensional wave equation: \( w_{xx} - w_{yy} = 0 \) to \( w_{st} = 0 \).