Due: Final Exam period (tuesday noon)

1. For the problem \(-\Delta u = f\) in bounded domain \(\Omega \subset \mathbb{R}^n\); \(u = g\) on \(\Gamma = \partial \Omega\), finish the proof of the following well-posedness result that was started in class: There exists \(C\) independent of \(\{f, g\}\) so that \(\|u\|_\infty \leq C\|\{f, g\}\|_\infty\). (We had proved it in the case \(f \equiv 0\).) (Note: The book gives a proof on p. 78, assuming that (3.4) has a solution. Try to skip this assumption, by finding a \(v\) that satisfies inequalities similar to (3.3) and (3.5) without (3.4).)

2. Derive a maximum principle for \(-\text{div}(k\nabla u) \leq 0\) in \(\Omega\) (bounded domain in \(\mathbb{R}^n\)); where \(k > 0\) is \(C^1(\bar{\Omega})\). (Hint: This is similar to proof on p. 519, but replace \(|x|^2\) by something else.)

3. Derive a maximum principle for \(-u'' + pu' \leq 0\), \(0 < x < 1\), where \(p\) is a continuous function.

4. Find the modified Green’s function that is needed to solve \(-u'' = f\), \(-1 < x < 1\), \(u'(1) = u(1), u'(-1) = -u(-1)\) when \(f\) satisfies appropriate solvability conditions.

5. p. 522 prob. 2.2 (Just answer the first part: Show that if on \(\Gamma_d, u \leq M\), where \(M \geq 0\), then \(u \leq M\) in \(\Omega\)).

6. Prove metrics are continuous (i.e., if \(u_n \to u\) and \(v_n \to v\) as \(n \to \infty\), then \(d(u_n, v_m) \to d(u, v)\) as \(n, m\) go to infinity).

7. Let \((K, d)\) be a compact metric space. (Each sequence in \(K\) has a convergent subsequence.) Prove
   (i) \((K \times K, \rho)\) is compact where \(\rho((u, v), (U, V)) = \max(d(u, U), d(v, V))\).
   (ii) if \(T : K \to K\) satisfies \(d(T(u), T(v)) < d(u, v)\) for all \(u, v\) in \(K\), then \(T\) is a contraction.
   (iii) show that this is false if \(K\) is not assumed to be compact.

8. For the nonlinear boundary value problem: \(-u'' + \lambda \cos u = f\), \(u(0) = u'(1) = 0\), determine a range of values for \(\lambda\) for which there exists a unique solution. (Assume \(f\) is continuous.)