Solvability conditions

Some useful theorems:

1. If H is s subspace of $\mathbb{R}^n$ then $(H^\perp)^\perp = H$.

2. If A is an $n \times m$ matrix then $(\text{Col}(A))^\perp = \text{Nul}(A^T)$.

Putting 1. and 2. together we get:

3. If A is an $n \times m$ matrix then $(\text{Col}(A)) = (\text{Nul}(A^T))^\perp$.

Also mentioned in the book:

4. If H is s subspace of $\mathbb{R}^n$ and $H = \text{span}(\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p)$ then w is in $H^\perp$ if and only if $w \cdot \vec{v}_j = 0$, $j = 1, 2, \ldots p$.

Earlier we had that

5. If A is an $n \times m$ matrix and $\vec{b}$ is in $\mathbb{R}^m$, then $A\vec{x} = \vec{b}$ has a solution (i.e., is consisent) if and only if $\vec{b}$ is in $\text{Col}(A)$.

Now consider the system

$$A\vec{x} = \vec{b}.$$

It is easy to find a basis $\{\vec{b}_1, \ldots \vec{b}_p\}$ for $\text{Nul}(A^T)$. Applying Theorem 4 to this situation, we obtain the *solvability conditions* : $\vec{b} \cdot \vec{b}_j$, $j = 1, 2 \ldots, p$. There will be at least one solution to $A\vec{x} = \vec{b}$ if and only if $\vec{b}$ satisfies all the solvability conditions.

Example: Find solvability conditions for $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 5 & 0 \\ -5 & -6 & 0 \end{bmatrix}$.

Solution: We have $A^T = \begin{bmatrix} 1 & 2 & 6 & -5 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A short calculation gives $\begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -7 \\ 6 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for $\text{Nul}(A^T)$. Thus the solvability conditions are $\vec{b} \cdot \begin{bmatrix} 4 \\ -5 \\ 1 \\ 0 \end{bmatrix} = \vec{b} \cdot \begin{bmatrix} -7 \\ 6 \\ 1 \\ 0 \end{bmatrix} = 0$.

Exercize: Find solvability conditions for $A\vec{x} = \vec{b}$, where

$$A^T = \begin{bmatrix} 0 & 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$