Math 166  Homework 4 Solutions

1) Indicate whether the series converges or diverges and give a reason for your conclusion.

(a) \[ \sum_{n=1}^{\infty} \frac{n^2 + 1}{n + n^3} \]

SOLUTION: Use limit comparison to the harmonic series.

\[ \lim_{n \to \infty} \frac{n^2 + 1}{n + n^3} / (1/n) = 1 \]

Since this is between 0 and \( \infty \) the series does the same as the harmonic series which DIVERGES.

(b) \[ \sum_{n=2}^{\infty} \frac{1000}{n \ln(n)} \] (Hint: use integral test)

SOLUTION: \( \int_{2}^{\infty} \frac{1000}{x \ln(x)^{1.1}} \, dx \) (let \( u = \ln x \), \( du = x^{-1} \, dx \)) = \( \int_{\ln 2}^{\infty} \frac{1000 \, du}{u^{1.1}} \), which CONVERGES since \( p > 1 \). Thus the series also converges. (Remark: to be rigorous, we should verify that \( \frac{1000}{x \ln(x)^{1.1}} \) is decreasing after some point. This follows if \( x \ln x \) is increasing after some point. But this is true since the derivative \( \ln x + 1 \) is positive for \( x > e \).)

(c) \[ \sum_{n=1}^{\infty} \frac{n^2 + n \cos n}{n + n^3} \]

SOLUTION: Use limit comparison test to the harmonic series again. Again you find \( \lim_{n \to \infty} \frac{(n^2 + n \cos n)}{(n + n^3)} / (1/n) = 1 \) and hence the series DIVERGES like the harmonic series does.

(d) \[ \sum_{n=1}^{\infty} \frac{n^6}{1.1^n} \]

SOLUTION: Use the ratio test.

\[ \lim_{n \to \infty} \frac{(n+1)^6}{1.1^{n+1}} = \lim_{n \to \infty} \frac{(n+1)^6}{n^6} 1.1 = \frac{1}{1.1} < 1 \]

Hence it CONVERGES by ratio test.
2) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

(a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n \tan^{-1} n}{n+2} \]

SOLUTION: Since \( \tan^{-1} n \to \pi/2 \), \( (\tan^{-1} n)/n \to 0 \) the series CONVERGES by alt.series test. To test for absolute convergence, use limit comparison test or comparison test to harmonic series. To use limit comparison test:

\[
\lim_{n \to \infty} \left( \frac{\tan^{-1} n}{n+2} \right)/\left( \frac{1}{n} \right) = \pi/2
\]

Hence the series does not converge absolutely. Thus series CONVERGES CONDITIONALLY.

(b) \[ \sum_{n=1}^{\infty} (-1)^n \tan(n^{-2}) \] (Hint: for abs. convergence, use limit comparison to a \( p \)-series)

SOLUTION: For convergence, apply alternating series test: \( \lim_{n \to \infty} \tan(n^{-2}) = 0 \). Thus the series converges. To test absolute convergence, compare to \( \sum_{n=1}^{\infty} \frac{1}{n^2} \).

\[
\lim_{n \to \infty} \left( \frac{\tan(n^{-2})}{n^2} \right) \text{ (let } s = 1/n^2 \text{)} = \lim_{s \to 0} \frac{\tan(s)}{s} = \lim_{s \to 0} \sec^2 s = 1
\]

by L’Hopital’s rule. Therefore series does the same as the \( p \)-series which converges \( (p = 2) \). Hence, Series CONVERGES ABSOLUTELY.

(c) \( 1 - \frac{7}{2\pi} + \frac{7^2}{3\pi} - \frac{7^3}{4\pi} + \ldots \)

SOLUTION: First put in summation notation: \( \sum_{n=1}^{\infty} \frac{(-7)^{n-1}}{n^n} \). Then apply ratio test

\[
\lim_{n \to \infty} \left| \frac{(-7)^n}{(n+1)^{n+1}} \cdot \frac{n^n}{(-7)^{n-1}} \right| = \lim_{n \to \infty} \frac{7}{n+1} \left( \frac{n}{n+1} \right)^n = 0
\]

Thus series CONVERGES ABSOLUTELY by ratio test. (Comment: the last limit above follows from easy from squeeze theorem: \( 0 \leq \frac{7}{n+1} \left( \frac{n}{n+1} \right)^n \leq \frac{7}{n+1} \). Since both 0 and \( \frac{7}{n+1} \) go to zero as \( n \to \infty \), so does the middle term.)