

A Martingale Central Limit Theorem

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We present a proof of a martingale central limit theorem (Theorem 2) due to McLeish (1974). Then, an application to Markov chains is given.

Lemma 1. For $n \geq 1$, let U_n, T_n be random variables such that

1. $U_n \rightarrow a$ in probability.
2. $\{T_n\}$ is uniformly integrable.
3. $\{|T_n U_n|\}$ is uniformly integrable.
4. $E(T_n) \rightarrow 1$.

Then $E(T_n U_n) \rightarrow a$.

Proof. Write $T_n U_n = T_n(U_n - a) + aT_n$. As $E[T_n] \rightarrow 1$, we need only show that $E[T_n(U_n - a)] \rightarrow 0$ to finish.

Since $\{T_n\}$ is uniformly integrable, we have $T_n(U_n - a) \rightarrow 0$ in probability. Also, both $T_n U_n$ and aT_n are uniformly integrable, and so the combination $T_n(U_n - a)$ is uniformly integrable. Hence, $E[T_n(U_n - a)] \rightarrow 0$. \square

A key observation for the following is the expansion,

$$\exp(ix) = (1 + ix) \exp\left(-\frac{x^2}{2} + r(x)\right)$$

where $|r(x)| \leq |x|^3$ for real x .

Theorem 1. Let $\{X_{nj} : 1 \leq j \leq k_n, n \geq 1\}$ be a triangular array of (any) random variables. Let $S_n = \sum_{1 \leq j \leq k_n} X_{nj}$, $T_n = \prod_{1 \leq j \leq k_n} (1 + itX_{nj})$, and $U_n = \exp\left(-\frac{t^2}{2} \sum_j X_{nj}^2 + \sum_j r(tX_{nj})\right)$. Suppose that

1. $E(T_n) \rightarrow 1$.
2. $\{T_n\}$ is uniformly integrable.
3. $\sum_j X_{nj}^2 \rightarrow 1$ in probability.
4. $\max_j |X_{nj}| \rightarrow 0$ in probability.

Then $E(\exp(itS_n)) \rightarrow \exp(-\frac{t^2}{2})$.

Proof. Let t be fixed. From conditions (3) and (4), bound

$$\begin{aligned} \left| \sum_j r(tX_{nj}) \right| &\leq |t|^3 \sum_j |X_{nj}|^3 \\ &\leq |t|^3 \max_j |X_{nj}| \sum_j X_{nj}^2 = o(1). \end{aligned}$$

Then,

$$\begin{aligned} U_n &= \exp\left(-\frac{t^2}{2} \sum_j X_{nj}^2 + \sum_j r(tX_{nj})\right) \\ &= \exp\left(-\frac{t^2}{2} + o(1)\right). \end{aligned}$$

This verifies condition (1) of Lemma 1 with $a = \exp(-\frac{t^2}{2})$.

Conditions (2) and (4) of Lemma 1 are our present conditions (2) and (1), respectively. Condition (3) of Lemma 1 follows from the fact

$$|T_n U_n| = |\exp(itS_n)| = \left| \exp(it \sum_j X_{nj}) \right| = 1.$$

Thus $E(\exp(itS_n)) = E(T_n U_n) \rightarrow \exp(-t^2/2)$. □

Theorem 2. Let $\{X_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ be a martingale difference array with respect to nested σ -fields $\{\mathcal{F}_{nj} : 1 \leq j \leq k_n, n \geq 1\}$, $\mathcal{F}_{nj} \subset \mathcal{F}_{nk}$ for $j \leq k$, such that

1. $\{\max_j |X_{nj}|\}$ is uniformly integrable
2. $E(\max_j |X_{nj}|) \rightarrow 0$
3. $\sum_j X_{nj}^2 \rightarrow 1$ in probability.

Then $S_n = \sum_j X_{nj} \Rightarrow N(0, 1)$ in distribution.

Proof. Define $Z_{n1} = X_{n1}$, and $Z_{nj} = X_{nj}I(\sum_{1 \leq r \leq j-1} X_{nr}^2 \leq 2)$ for $2 \leq j \leq k_n$ and $n \geq 1$. Then $\{Z_{nj} : 1 \leq j \leq k_n, n \geq 1\}$ is also martingale difference array with respect to $\{\mathcal{F}_{nj}\}$ because

$$E(Z_{nj} | \mathcal{F}_{n(j-1)}) = I\left(\sum_{r \leq j-1} X_{nr}^2 \leq 2\right) E(X_{nj} | \mathcal{F}_{n(j-1)}) = 0.$$

Let now $J = \inf\{j : \sum_{1 \leq r \leq j} X_{nr}^2 > 2\} \wedge k_n$. Then,

$$\begin{aligned} P(X_{nr} \neq Z_{nr} \text{ for some } r \leq k_n) &= P(J \leq k_n - 1) \\ &\leq P\left(\sum_{r \leq k_n} X_{nr}^2 > 2\right) \rightarrow 0 \end{aligned} \quad (1)$$

from the third assumption.

It is also to easy that the variables $\{Z_{nj}\}$ satisfy the conditions of the Theorem 2.

We now show that $\{Z_{nj}\}$ satisfies the conditions of Theorem 1. Let $T_n = \prod_{j \leq k_n} (1 + itZ_{nj})$. Since $|(1 + itx)|^2 = (1 + t^2x^2) \leq \exp(t^2x^2)$, we have

$$\begin{aligned} |T_n| &= \prod_{1 \leq r \leq J-1} (1 + t^2 X_{nr})^{1/2} (1 + t|X_{nJ}|) \\ &\leq \exp((t^2/2) \sum_{1 \leq r \leq J-1} X_{nr}^2) (1 + t|X_{nJ}|) \\ &\leq \exp(t^2) (1 + t \max_j |X_{nj}|). \end{aligned}$$

As $\{\max_j |X_{nj}|\}$ is uniformly integrable by assumption (1), we have $\{T_n\}$ is uniformly integrable. Also, as $\{Z_{nr}\}$ is a martingale difference array, we have by successive conditioning that $E(T_n) = 1$. Hence, conditons (1) and (2) of Theorem 1 for $\{Z_{nj}\}$ are met.

Clearly conditions (3) and (4) of Theorem 1 also hold for the array $\{Z_{nj}\}$ in view of (1) and assumptions (2) and (3).

Thus all the conditions of Theorem 1 hold, and we conclude $\sum_{r \leq k_n} Z_{nr} \rightarrow N(0, 1)$. But, by (1), we have then that $\sum X_{nr} \rightarrow N(0, 1)$ also. \square

For some applications, the following corollary of Theorem 2 is convenient.

Theorem 3. Let $\{Z_j : j \geq 1\}$ be a stationary ergodic sequence such that $\sigma^2 = E[Z_1^2] < \infty$, and $E[Z_{n+1} | \mathcal{F}_n] = 0$ where $\mathcal{F}_n = \sigma\{Z_j : j \leq n\}$. Then, we

have

$$Y_n = \frac{1}{\sqrt{n}}[Z_1 + \cdots + Z_n] \Rightarrow N(0, \sigma^2).$$

Proof. Let $X_{nj} = Z_j/\sqrt{n}$ and $\mathcal{F}_{nj} = \mathcal{F}_j$ for $1 \leq j \leq n$ and $n \geq 1$. Then, $\{X_{nj}\}$ is a martingale difference array with respect to $\{\mathcal{F}_{nj}\}$. The proof follows directly from Theorem 2. \square

We now present an application of Theorem 3 to finite state Markov chains in discrete time.

Application. Let Σ be a finite state space with r letters, $|\Sigma| = r$. Let $\{X_i : i \geq 1\}$ be an ergodic Markov chain on Σ with transition matrix P starting under the stationary measure π .

Let also $f : \Sigma \rightarrow R$ be a mean-zero function with respect to π , $E_\pi[f] = 0$. Consider now the sum $S_n = \sum_{i=1}^n f(X_i)$.

The aim of this application is to show that S_n/\sqrt{n} converges in distribution to $N(0, \sigma^2)$ for some $\sigma^2 < \infty$ with the help of Theorem 3.

A preliminary lemma will be useful. Let I_r be the $r \times r$ identity matrix. Also note that f can be represented as a vector, $f = \langle f(i) : i \in \Sigma \rangle \in R^r$.

Lemma 2. There is a function $u : \Sigma \rightarrow R$ such that $f = (I_r - P)u$.

Proof. Write

$$R^r = \text{Null}(I - P^*) \oplus \text{Range}(I - P)$$

where P^* is the adjoint of P . Then, as $\pi[I - P] = 0$, and π is unique, we have

$$\text{Null}(I - P^*) = \{c\pi : c \in R\},$$

a one-parameter space. However, since $E_\pi[f] = 0$ and so $f \perp \pi$, we must have $f \in \text{Range}(I - P)$. \square

We now approximate S_n/\sqrt{n} by a martingale. For $n \geq 1$, define

$$M_n = \sum_{i=1}^n [u(X_i) - (Pu)(X_{i-1})] \quad \text{and} \quad \mathcal{F}_n = \sigma\{X_i : 1 \leq i \leq n\}.$$

From the Markov property, the conditional expectation, $E[u(X_i)|\mathcal{F}_{i-1}] = (Pu)(X_{i-1})$. Therefore, $\{M_n\}$ is a martingale difference sequence with respect to $\{\mathcal{F}_n\}$.

Write

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{i=1}^n f(X_i) &= \frac{1}{\sqrt{n}} \left[\sum_{i=1}^n u(X_i) - \sum_{i=1}^n (Pu)(X_i) \right] \\ &= \frac{M_n}{\sqrt{n}} + \frac{(Pu)(X_0) - (Pu)(X_n)}{\sqrt{n}}. \end{aligned}$$

As u is bounded, the error in the martingale approximation vanishes,

$$[(Pu)(X_0) - (Pu)(X_n)]/\sqrt{n} \rightarrow 0.$$

We now compute the variance σ^2 :

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} E_\pi[M_n^2] &= E_\pi[(u(X_1) - (Pu)(X_0))^2] \\ &= E_\pi[u^2 - (Pu)^2] \\ &= \sigma^2. \end{aligned}$$

As long as f is non-constant, u is non-constant and $\sigma^2 > 0$. Also, as u is bounded, $\sigma^2 < \infty$.

Hence, by Theorem 3, we have $S_n/\sqrt{n} \Rightarrow N(0, \sigma^2)$. □

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References

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