Diffusive Limit of a Tagged Particle in Asymmetric Simple Exclusion Processes

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Abstract

Invariance principles are proved in diffusive scale for the centered position of a tagged particle in the simple exclusion process with asymmetric nonzero drift jump probabilities in dimensions $d \geq 3$. The method of proof is by martingale techniques which rely on the observation that symmetric random walk is transient in high dimensions.

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1.Introduction.

In the study of interacting particle systems, an important step is the behavior of a tagged particle as it moves among the other particles which could be viewed as indistinguishable among themselves. An initial step is to understand the behaviour of a tagged particle in equilibrium, in particular the fluctuation behaviour. The simple exclusion processes are a natural class of models where one can study this problem. In the case of simple exclusion processes on Z^d when the underlying random walk has a symmetric distribution, this question is answered in [A] for $d = 1$ nearest-neighbor walks, and in [KV] for all other distributions and dimensions. It was generalized to the case of random walks with mean 0 in [V]. When the mean is non-zero, the only case studied previously is the asymmetric nearest-neighbor $d = 1$ case in [K]. Here we consider the general case where the mean is not zero. We need to assume however that *the dimension d of the lattice Z^d is at least 3.*

If X is a countable set and $p(x, y)$ is the transition probability matrix of a Markov chain, then one can define the simple exclusion process on X with transition probabilities $p(\cdot, \cdot)$ on X. This is actually a Markov process whose state space is the set $\Omega(X)$ of all possible subsets of X. The set $A \subset X$ signifies that there are particles present at every point of A and that the sites in $X \backslash A$ are empty. One can also view $\Omega(X)$ as the set of maps $\eta : X \to \{0, 1\}$ where $\eta(x) = 1$ means that $x \in A$ or that there is a particle at the site x. $\eta(x) = 0$ signifies that the site is empty. Each particle waits for an exponential time and then jumps to a new site or at least tries to. The new site y is selected with probability $p(x, y)$ where the current site is x. If the site y is empty then the jump is completed and things start afresh. If the site is occupied with a particle already there the jump is forebidden and the original particle remains at x. Again things start afresh. All the particles are doing this simultaneously and because we are dealing with continuous time there will be no ties to resolve. However it requires some work to make sure that the process is well defined especially if there are an infinite number of particles to begin with.

Suppose $u : \Omega \to R$ is a function that depends only on a finite number of cordinates, i.e on $\{\eta(x) : x \in F\}$ for some finite set F, then the infinitesimal generator of the process is defined by

$$(\mathcal{L}u)(\eta) = \sum_{x,y} \eta(x)(1 - \eta(y))p(x, y)[u(\eta^{x,y}) - u(\eta)]$$

where

$$\begin{cases} \eta^{x,y}(z) = \eta(z) \ \text{ if } \ z \neq x \ \text{ or } \ y \\ \eta^{x,y}(x) = \eta(y) \\ \eta^{x,y}(y) = \eta(x). \end{cases}$$

Under some mild conditions on $p(\cdot,\cdot)$ [for instance it is sufficient to assume that $\sup_y \sum_x p(x,y) < \infty$] there is a well defined stocahstic process starting from any initial configuration in Ω. The details can be found in [L]. This is referred to as the simple exclusion process on X with transition probability $p(\cdot,\cdot)$. A special case of particular interest is when $p(\cdot,\cdot)$ is doubly stochastic, i.e

$$\sum_x p(x,y) = 1, \quad \text{for all} \quad y$$

in addition to

$$\sum_y p(x,y) = 1, \quad \text{for all} \quad x.$$

In this case the process is always well defined. An even special case is when $p(\cdot,\cdot)$ is symmetric. This case is referred to as symmetric simple exclusion.

In the doubly stochastic case the uniform measure on X is a σ-finite invariant measure for the Markov chain. One can verify that for any $0 \le \rho \le 1$, the Bernoulli product measure P_ρ on Ω defined by $P_\rho[\eta(x) = 1] = \rho$ for every x, with $\{\eta(x)\}$ being mutually independent for different x, is an ergodic invariant measure for the evolution on Ω with generator L. If $p(\cdot,\cdot)$ is symmetric, the evolution is reversible with respect to each P_ρ and has for its Dirichlet form the quantity

$$D_\rho(u) = \frac{1}{2}\int_\Omega \sum_{x,y} p(x,y)\ [u(\eta^{x,y}) - u(\eta)]^2\ dP_\rho.$$

Our interest is mainly in the case where $X = Z^d$ for some $d \ge 3$ and $p(x,y) = p(y-x)$ for some probability distribution $p(\cdot)$ on Z^d. We can assume without loss of generality that $p(0) = 0$. We are interested in the case when

$$m = \sum_x x p(x) \neq 0. \tag{1.1}$$

We shall assume for simplicity that $p(x) = 0$ outside a finite set F although it will not matter that much. We shall start the process in equilibrium, that is to say, with the initial distribution being some P_ρ but conditioned to have a particle at the origin 0, which will be tagged. As the system evolves we wish to follow the trajectory of the tagged particle. It is convenient to change our coordinates in Z^d so that the tagged particle is always seen at the origin. In other words our description of the current state consists of the position of the tagged particle which we denote by z and the environment seen from the tagged partcle which can be viewed as a point in $\Omega_0 = \{\eta : Z^d \backslash \{0\} \to \{0,1\}\}$. There are two types of motions. When an untagged particle jumps it is from some x to a y, neither of which can be 0. When the tagged particle jumps from 0 to an empty site x, the origin shifts with it, so what we see is a shift of the environment by $-x$. The tagged particle at 0 is not part of the shift and so we always end up with $-x$ being empty. In other words, if we define on the set $\eta(x) = 0$, the map τ_x

$$\begin{cases} (\tau_x\eta)(y) = \eta(x+y) \quad \text{for} \quad y \neq 0 \quad \text{or} \quad -x \\ (\tau_x\eta)(-x) = 0 \end{cases}$$

then the generator of our process is given by $\mathcal{L} = \mathcal{L}^{sh} + \mathcal{L}^{ex}$ where

$$(\mathcal{L}^{sh}u)(z,\eta) = \sum_x (1-\eta(x))p(x)[u(z+x,\tau_x\eta) - u(z,\eta)]$$

and

$$(\mathcal{L}^{ex}u)(z,\eta) = \sum_{x,y\neq 0} (u)\eta(x)(1-\eta(y))p(y-x)[u(z,\eta^{x,y}) - u(z,\eta)].$$

The environment by itself is a Markov process and the generator is given by $L = L^{sh} + L^{ex}$ where

$$(L^{sh})(\eta) = \sum_x (1-\eta(x))p(x)[u(\tau_x\eta) - u(\eta)] \tag{1.2}$$

$$(L^{ex})(\eta) = \sum_{x,y\neq 0} \eta(x)(1-\eta(y))p(y-x)[u(\eta^{x,y}) - u(\eta)]. \tag{1.3}$$

We have adopted the convention that the generators associated with the original process are denoted by script $\mathcal{L}$ while those associated with the environmental process by L. The Bernoulli product measure P_ρ restricted to points in $Z^d \backslash \{0\}$ is an ergodic invariant measure for the environment process (see Proposition 3 of [S]). An elementary computation shows that

$$Lz = g(\eta) = \sum_x xp(x)(1-\eta(x)) \tag{1.4}$$

so that

$$z(t) - z(0) - \int_0^t g(\eta(s))ds = M(t)$$

is a martingale with stationary increments. One can see almost surely [S],

$$\lim_{t\to\infty} \frac{z(t)}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t g(\eta(s))ds = \int_{\Omega_0} g(\eta)dP_\rho = m(1-\rho).$$

Let

$$\xi(t) = \frac{1}{\sqrt{t}}(z(t) - m(1-\rho)t); \tag{1.5}$$

in this article we prove a functional central limit theorem for ξ. As remarked earlier the symmetric case was considered in [A] for nearest-neighbor $d = 1$ walks, and [KV] generally. The more general case $m = 0$ was covered in [V]. Also, the $d = 1$ asymmetric nearest neighbor $m \neq 0$ case is done in [K]. We are now interested in the case when $m \neq 0$ in high dimensions which require different methods. Our main result is the following Theorem:

Theorem 1.1 . *The distributions of* $\frac{1}{\sqrt{\alpha}}(z(\alpha t) - m(1-\rho)\alpha t)$ *converge as* $\alpha \uparrow \infty$ *in Skorohod space to a nondegenerate Brownian motion with covariance* $\mathbf{C}(\rho)$.

In next section, we outline the proof of this theorem. Our proof follows the approach of [KV] to consider the environment process seen from the tagged particle and to explore the associated martingale. The key step of this approach is an estimate of a resolvent equation. Some of ideas for solving resolvent equation are related to the methods for solving the fluctuation-dissipation equation of the hydrodynamical limit of the simple exclusion processes [LY] and lattice gases [EMY]. Notice that both the fluctuation-dissipation equation or the resolvent equation require estimates on the Green's function of the generator, though in different contexts. More technical comments can be found at the end of next section.

2. Outline of the Proof.

The tagged particle process has a lot of martingales associated with it. They are of the form

$$M_{x,y}(t) = N_{x,y}(t) - \int_0^t p(y-x)\eta(s,x)(1-\eta(s,y))ds$$

corresponding to the number $N_{x,y}(t)$ of jumps of untagged particles from x to y or

$$M_x(t) = N_x(t) - \int_0^t p(x)(1 - \eta(s,x))ds$$

corresponding to the number of jumps of size x for the tagged particle. The quadratic variations are given by

$$[dM_{x,y}(t)]^2 = p(y-x)\eta(t,x)(1-\eta(t,y))dt, \quad [dM_x(t)]^2 = p(x)(1-\eta(t,x))dt.$$

These are the basic martingales and every other martingale is a combination of these. For example, the position $z(t)$ of the tagged particle satisfies

$$z(t) = \sum_x xN_x(t) = \sum_x xM_x(t) + \int_0^t \sum_x xp(x)(1-\eta(s,x))ds.$$

Subtracting the term $m(1-\rho)t$, we have

$$\xi(t) = z(t) - m(1-\rho)t = \xi^{(1)}(t) + \int_0^t g(\eta(s))ds \tag{2.1}$$

where

$$\xi^{(1)}(t) = \sum_x xM_x(t) \tag{2.2}$$

and g is the vector valued function given by (1.4), i.e., $g(\eta) = \sum_x xp(x)(\rho - \eta(x))$.

The problem now reduces to proving the central limit theorem for the centered additive functional

$$A(t) = \int_0^t g(\eta(s))ds.$$

We would like to do it by the martingale method, which means finding square integrable martingales $\xi^{(2)}(t)$ such that

$$A(t) = \xi^{(2)}(t) + B(t)$$

with $B(\cdot)$ becoming negligible under rescaling. Then the central limit theorem for square integrabe martingales with stationary increments applies to the sum $\xi^{(1)}(t) + \xi^{(2)}(t)$ and this will establish the result. We may write the generator of the process which describes the environment of the tagged particle as the sum of its symmetric and antisymmetric pieces,

$$L = L_{sym} + L_{skew}.$$

Associated with the symmetric part, we have the Dirichlet form

$$D^{\rho}(u) = D^{ex,\rho}(u) + D^{sh,\rho}(u) \tag{2.3}$$

where

$$D^{ex,\rho}(u) = \frac{1}{2}\int_{\Omega_0} \sum_{x,y\neq 0} a(y-x)[u(\eta^{x,y}) - u(\eta)]^2 dP_\rho \tag{2.4}$$

$$D^{sh,\rho}(u) = \frac{1}{2}\int_{\Omega_0} \sum_{x} a(x)(1-\eta(x))[u(\tau_x\eta) - u(\eta)]^2 dP_\rho \tag{2.5}$$

with $a(x) = \frac{1}{2}[p(x) + p(-x)]$. The associated Dirichlet norm is $\|.\|_1 = \sqrt{D_\rho(u)}$. The dual norm $\|.\|_{-1}$ is defined by

$$\|g\|_{-1} = \inf\big[C : \big|\int g\ u\ dP_\rho\big| \leq C\|u\|_1 \text{ for all } u \text{ with } \|u\|_1 < \infty\big].$$

Define $\mathbf{H}_1$ and $\mathbf{H}_{-1}$ as the Hilbert spaces generated by local functions with respect to the norms $\|.\|_1$ and $\|.\|_{-1}$. We shall drop the index ρ which is fixed through the whole paper.

Outline of the Proof of Theorem 1.1:

Step 1: H_{-1} Estimate and Tightness. Our first step is to show that for the function g defined in (1.1), $\|g\|_{-1} < \infty$. This follows from the next lemma.

Lemma 2.1 . *Let f be a mean zero local function, i.e. a function that depends only on finite number of coordinates. Then for $d \geq 3$, there is a bound*

$$| < f, v > | \leq C_f \|v\|_{-1}$$

Lemma 2.1 will be proved in section 3. Recall that $\sqrt{t}\,\xi$ is a sum of a martingale $\xi^{(1)}$ and $A(t)$. The martingale $\xi^{(1)}$ satisfies Doob's inequality. Hence the tightness of ξ in Skorohod space is deduced from the following general theorem.

Theorem 2.2 . *Suppose we have a Markov process on a finite state space with a generator*

$$(\mathcal{A}f)(x) = \sum_y c(x,y)[f(y) - f(x)]$$

with an ergodic invariant probability measure $p(x)$ and the reversed process has the generator

$$(\mathcal{A}^* f)(x) = \sum_y c^*(x,y)[f(y) - f(x)]$$

which is the adjoint of $\mathcal{A}$ in $L_2[p(\cdot)]$. Let $S = \frac{1}{2}[\mathcal{A}+\mathcal{A}^*]$ be the symmetrized operator. Let $Su = f$ with

$$< -Su, u >= D(u) = \|f\|_{-1}^2 = C. \tag{2.6}$$

Finally let P refer to the stationary Markov Process with generator $\mathcal{A}$ and invariant measure $p(\cdot)$ as marginals. Then for all $T \geq 0$

$$E^P\Big\{\sup_{0\leq t\leq T} \big|\int_0^t f(x(s))ds\big|^2\Big\} \leq 16CT$$

with the same constant C as in (2.6).

Proof: We deal with both the forward and backward filtrations.

$$u(x(t)) - u(x(s)) - \int_s^t (\mathcal{A}u)(x(\sigma))d\sigma = M^+(t) - M^+(s)$$

is a Martingale adapted to the forward filtration for $t \geq s$.

$$u(x(s)) - u(x(t)) + \int_s^t (\mathcal{A}^*u)(x(\sigma))d\sigma = M^-(s) - M^-(t)$$

ia martingale adapted to the backward filtration for $s \leq t$. In any case

$$\int_s^t f(x(\sigma)d\sigma = \frac{1}{2}[M^+(t) - M^+(s) + M^-(s) - M^-(t)].$$

Since

$$E^P\big\{[M^+(t) - M^+(s)]^2\big\} = E^P\big\{[M^-(t) - M^-(s)]^2\big\} = C|t-s|$$

the theorem is a consequence of Doob's inequality applied separately to the two martingales. The estimate has a universal constant and is therefore valid always. $\otimes$

Step 2: Resolvent Estimate. Consider the resolvent equation

$$\lambda u_\lambda - Lu_\lambda = f. \tag{2.7}$$

The key input for the proof of Theorem 1.1 is the following estimate of the resolvent equation to be proved in section 6.

Theorem 2.3 . *For any local f with mean zero and $d \geq 3$,*

$$\sup_\lambda \|Lu_\lambda\|_{-1} < \infty, \quad \sup_\lambda \|\lambda u_\lambda\|_{-1} < \infty. \tag{2.8}$$

Step 3: Approximation Via the Resolvent Equation.

Theorem 2.4 . *If f is a local function, then for the solution u_λ of the resolvent equation we have*

$$\lim_{\lambda \to 0} \|u_\lambda - w\|_1 = 0 \tag{2.9}$$

for some $w \in \mathbf{H}_1$ and

$$\lim_{\lambda \to 0} \lambda \|u_\lambda\|_0^2 = 0 \tag{2.10}$$

Proof: Multiplying the resolvent equation by u_λ and integrating we get

$$\lambda \|u_\lambda\|_0^2 + \|u_\lambda\|_1^2 = < u_\lambda, f > . \tag{2.11}$$

By the $\mathbf{H}_1$ estimate of f in Theorem 2.1, $\|u_\lambda\|_1 < \infty$. Hence we can choose a subsequence $\lambda_n \to 0$ such that along this subsequence $u_{\lambda_n} = u_n$ has a weak limit w in the Dirichlet Space $\mathbf{H}_1$. Because $\lambda \|u_\lambda\|_0^2$ is bounded, $\lambda u_\lambda \to 0$ in $L_2 = \mathbf{H}$ and consequently, from (2.8), converges to 0 in $\mathbf{H}_{-1}$. Together with the resolvent equation, this implies that Lu_n converges weakly to $-f$ in $\mathbf{H}_{-1}$. By standard functional analysis there are convex combinations v_n of $u_1, \cdots, u_n$ such that v_n and Lv_n converge strongly to w and $-f$ in $\mathbf{H}_1$ and $\mathbf{H}_{-1}$ respectively. It is easy to see, from (2.8), that $\|v_n\|_1 \to \|w\|_1$ and $< v_n, Lv_n >$ converges to $- < w, f >$, thus proving

$$< w, f > = \|w\|_1^2.$$

From (2.11), we have in particular, finding now the limit w along a subsequence λ_n which leads to $\limsup < u_\lambda, f >$,

$$\limsup_{\lambda \to 0} \left[\ \lambda \|u_\lambda\|_0^2 + \|u_\lambda\|_1^2 \ \right] \leq < w, f > . \tag{2.12}$$

Since $\| \cdot \|_1$ is lower semicontinuous and $u_\lambda \to w$ weakly,

$$\|w\|_1^2 \leq \limsup_{\lambda \to 0} \|u_\lambda\|_1^2.$$

From (2.12) and the relation $< w, f > = \|w\|_1^2$, we conclude that equality holds in (2.12). This in turns implies that u_λ converges to w strongly and concludes the proof of Theorem 2.4. $\otimes$

We now have the following martingale decomposition theorem.

Theorem 2.5 . *There is a square integrable martingale $M(t)$ with stationary increments, and an additive functional $\Omega(t)$ such that*

$$\int_0^t f(\eta(s))ds = M(t) + \Omega(t)$$

with

$$T^{-1}EM(T)^2 \leq \|f\|_{-1}^2 \tag{2.13}$$

and

$$\lim_{T\to\infty} \frac{1}{T} E\left[[\Omega(T)]^2\right] = 0 \ . \tag{2.14}$$

Proof: The proof is based on Theorems 2.4. From Ito's formula

$$u_\lambda(\eta(t)) - u_\lambda(\eta(0)) = \int_0^t Lu_\lambda(\eta(s))ds + M_\lambda(t)$$

where M_λ denotes the martingale part with quadratic variation given by $EM_\lambda(t)^2 = \|u_\lambda\|_1^2$. From the resolvent equation (2.7),

$$\int_0^t f(\eta(s))ds = M_\lambda(t) + \Omega_\lambda(t)$$

where

$$\Omega_\lambda(t) = u_\lambda(\eta(t)) - u_\lambda(\eta(0)) - \int_0^t \lambda u_\lambda(\eta(s))ds \ .$$

By Theorem 2.4, the martingale part converges, as $\lambda \to 0$, to some limit that we call $M(t)$. Clearly, $M(t)$ satisfies the estimate (2.13). Hence we obtain

$$\int_0^t f(\eta(s))ds = M(t) + \Omega(t)$$

where

$$\Omega(t) = \Omega_\lambda(t) + [M_\lambda(t) - M(t)] \ .$$

If we pick $\lambda = \frac{1}{T}$ then together with (2.10)

$$\frac{1}{T}E^P\left[(\ \Omega(T)\ )^2 \right] \leq \frac{12}{T}\|u_{\frac{1}{T}}\|_0^2 + 4E^P\left[\ [M_{\frac{1}{T}}(1) - M(1)]^2\ \right] \to 0$$

as $T \to \infty$. This concludes the proof. $\otimes$

We can now apply Theorem 2.5 to our setting with f taken to be the function g defined in (1.4). Hence

$$\xi(t) = \xi^{(1)}(t) + \xi^{(2)}(t) + \Omega(t) \tag{2.15}$$

where $\xi^{(1)}$ is defined in (2.2) and $\xi^{(2)}(t)$ is the martingale obtained from applying Theorem 2.5 and $\Omega(t)$ is the error term satisfying (2.14). Hence up to a negligible error we have two square integrable martingales $\xi_1(t)$ and $\xi_2(t)$ with stationary increments adapted to the environment process. From the martingale convergence theorem (see Theorem 3.2 of [H] where condition (b) can be checked in this situation), $\xi(\alpha t)/\sqrt{\alpha}$, as $\alpha \uparrow \infty$, converges to a Brownian motion with the covariance matrix $\mathbf{C}(\rho)$ characterized by

$$< \mathbf{C}(\rho)a, a >= E^{P_\rho}\big\{ < \xi_1(1) + \xi_2(1), a >^2 \big\}\,. \tag{2.16}$$

Hence our final task is to estimate the variance.

Step 5: Positivity of Variance. For upper bounds of the variance we can estimate each one separately, where as for lower bounds we have to worry about possible cancellations. We start with the upper bound. From the definition of $\xi^{(1)}$ in (2.2) we have immediately

$$E^{P_\rho}\big\{ < \xi_1(1), a >^2 \big\} = (1-\rho)\sum_x p(x) < x, a >^2\,. \tag{2.17}$$

An upper bound of $\xi^{(2)}$ can also be easily obtained from (2.13)

$$E\big\{ < \xi_2(1), a >^2 \big\} \le C\rho(1-\rho)\sum_x p(x) < x, a >^2\,. \tag{2.18}$$

Therefore, we have the upper bound:

Theorem 2.6 .

$$< \mathbf{C}(\rho)a, a >\le C(1-\rho)\sum_x p(x) < x, a >^2\,.$$

The lower bound is stated as the following Theorem to be proved in the section 7.

Theorem 2.7 .

$$< \mathbf{C}a, a >\ge C_3(1-\rho) < a, a >\,.$$

We have based our proof on estimating the resolvent equation (2.7) with the key estimate (2.8). Alternatively, we can so choose to base the proof on the following estimate: *For any $\epsilon > 0$ there is a local function u_ε such that*

$$\|Lu_\varepsilon - f\|_{-1} \le \epsilon.$$

From this estimate, we can obtain a result analagous to Theorem 2.4 ([LY] or sect. 6 of [JY]) and thus the martingale decomposition theorem 2.5. This was the estimate established in [LY, EMY] for the fluctuaiton-dissipation equation. In a sense these two estimates on the Green's functions can be replace each other in many contexts. See sect. 6 of the lecture notes [JY].

3. Estimates Related to Simple Exclusion Processes.

Simple exclusion processes describe dynamics of infinitely many simple random walks on $\mathbb{Z}^d$ with the exclusion that no two particles are allowed to occupy the same site. The simple random walk on $\mathbb{Z}^d$ has the generator given by

$$(\boldsymbol{A} f)(x) = \sum_y p(y-x)[f(y) - f(x)].$$

The corresponding symmetric generator is

$$(S f)(x) = \sum_y a(y-x)[f(y) - f(x)] \tag{3.1}$$

where

$$p(x) = a(x) + b(x),$$

$a(\cdot)$ and $b(\cdot)$ being respectively the symmetric and asymmetric components of $p(\cdot)$. Note that

$$|b(x)| = \frac{1}{2}|p(x) - p(-x)| \le \frac{1}{2}[p(x) + p(-x)] = a(x).$$

The Dirichlet form is

$$\mathbb{D}(f) = \frac{1}{2}\sum_{x,y} a(y-x)[f(y) - f(x)]^2. \tag{3.2}$$

Since we are mainly interested in the tagged particle process, the state space is $\mathbb{Z}^d \setminus \{0\}$ rather than $\mathbb{Z}^d$. The random walk on $\mathbb{Z}^d \setminus \{0\}$ can be viewed as the random walk on $\mathbb{Z}^d$ with into the origin disallowed. The main result in this section was obtained in collaboration with C. Landim and was reported in [LY] with a purely analytic proof. Here we give a more probabilistic proof that applies in a more general context.

We can consider the general setting of an irreducible transient Markov process on a countable state space X which is symmetric with respect to the counting or uniform measure on X. The generator and the Dirichlet form are given by (3.1) and (3.2) with obvious interpretation. There is a subset $E \subset X$ that is excluded and transitions into E are disallowed. On the state space $Y = X \backslash E$ we have the induced generator

$$(\tilde{S}\, u)(x) = \sum_{y \in Y} a(x,y)[u(y) - u(x)]$$

for $x \in Y$. The corresponding Dirichlet form is

$$\tilde{I\!\!D}\,(u) = \frac{1}{2} \sum_{x,y \in Y} a(x,y)[u(y) - u(x)]^2.$$

For the special case of a random walk, E consists of just the origin. We shall keep the set E general because we want to apply it to other settings as well. Since random walk is transient for dimension $d \geq 3$, we shall assume that the basic process on X is transient, which means that the Green's function exists i.e.

$$g(x,y) = \int_0^\infty p(t,x,y)dt < \infty \quad \text{for all} \quad x, y \in X$$

where $p(t,x.y)$ is the transition probability function. The Green's function $g(x,y)$ satisfies

$$g(x,y) = g(y,x) \leq g(x,x) \quad \text{for all} \quad x, y \in X$$

and

$$(S\, g(x,\cdot))(y) = \sum_z [g(x,z) - g(x,y)] a(y,z) = -\delta_x(y) \quad \text{for all} \quad x, y \in X$$

where $\delta_x(y) = 1$ if $x = y$ and 0 otherwise.

We want to show that the induced process on Y with generator $\tilde{S}$ is again transient and compare its Green's function $\tilde{g}(x,y)$ to the original Green's function $g(x,y)$. We will assume that the probability

$$\theta(x) = P_x\{x(t) \ \text{ visits } \ E \ \text{ for } \ \text{some } \ t \geq 0\} < 1$$

for each $x \in Y$. This is clearly satisfied for the random walk with $E = \{0\}$.

It is well known that in the transient case there is an estimate of the form

$$|u(x)| \leq C(x)\sqrt{D(u)}$$

valid uniformly for all functions u that vanish outside a finite set and by completion for all functions that belong to the Dirichlet space $\mathbf{H}_1$, which we recall was defined as the completion of the space of finitely supported functions under the norm $\|u\|_1 = \sqrt{D(u)}$. A precise estimate on the constant $C(x)$ is given in the following lemma.

Lemma 3.1 . *Suppose $V(x)$ is a nonnegative compactly supported function. Then*

$$\sum_x u^2(x)V(x) \le \sup_x [\sum_y g(x,y)V(y)]\, \mathbb{D}(u)$$

for all u. In particular taking V to be 1 at x and 0 elsewhere

$$|u(x)| \le \sqrt{g(x,x)}\sqrt{\mathbb{D}(u)}.$$

Proof: Let us define $W(x) = \sum_y g(x,y)V(y)$ and $C = \sup_x W(x)$. Since $0 \le W \le C$ and $V \ge 0$,

$$\begin{aligned}
\sum_x u^2(x)V(x) &\le C \sum_x u^2(x)\frac{V(x)}{W(x)} \\
&= -C\sum_x \frac{u^2(x)}{W(x)}(\mathcal{A}W)(x) \\
&= \frac{C}{2}\sum_{x,y}[\frac{u^2(y)}{W(y)} - \frac{u^2(x)}{W(x)}][W(y) - W(x)]a(x,y) \\
&= \frac{C}{2}\sum_{x,y}[u^2(y) + u^2(x) - u^2(x)\frac{W(y)}{W(x)} - u^2(y)\frac{W(x)}{W(y)}]a(x,y) \\
&\le \frac{C}{2}\sum_{x,y}[u^2(y) + u^2(x) - 2u(x)u(y)]a(x,y) = C\mathbb{D}(u).
\end{aligned}$$

$\bigotimes$

Notice that $\theta(x)$ solves the equation

$$\begin{cases} (S\,\theta)(x) = \sum_{y\in X} a(x,y)[\theta(y) - \theta(x)] = 0 \quad \text{for} \quad x \in Y \\ \\ \theta(x) = 1 \quad \text{for} \quad x \in E. \end{cases}$$

Therefore

$$(\tilde{S}\,\theta)(x) = -\sum_{y\in E} a(x,y)[\theta(y) - \theta(x)] = -\sum_{y\in E} a(x,y)[1 - \theta(x)] = -A(x)(1-\theta(x))$$

where

$$A(x) = \sum_{y\in E} a(x,y). \tag{3.3}$$

Because of irreducibility $A(x)$ cannot vanish identically on Y. Thus $\theta(x)$ is a bounded nonconstant superharmonic function and this makes the process transient. We will actually assume that

$$\sup_{x\in Y} \theta(x) = \beta < 1. \tag{3.4}$$

The following lemma is a quantified version of the transience.

Lemma 3.2 . *For the Green's function $\tilde{g}(x,y)$, we have*

$$\sum_{y\in Y} \tilde{g}(x,y)A(y) \le \frac{\theta(x)}{(1-\beta)}.$$

Proof: Since

$$(\tilde{S}\,\theta)(x) \ge (1-\beta)A(x),$$

the lemma follows from the maximum principle. $\otimes$

Lemma 3.3 . *Let $U \ge 0$ be supported on Y, that is, vanish on E. Then*

$$\begin{aligned} \sup_{x\in Y} \sum_{y\in Y} \tilde{g}(x,y)\; U(y) &\le \frac{1}{(1-\beta)} \sup_{x\in X} \sum_{y\in X} g(x,y)U(y) \\ &= \frac{1}{(1-\beta)} \sup_{x\in Y} \sum_{y\in Y} g(x,y)U(y). \end{aligned} \tag{3.5}$$

Furthermore, for all $x, y \in Y$,

$$\tilde{g}(x,y) \le g(x,y) + \frac{1}{(1-\beta)} g(y,y)\theta(x) \tag{3.6}$$

and

$$\tilde{g}(x,x) \le \frac{1}{(1-\beta)} g(x,x). \tag{3.7}$$

Proof: Let

$$W(x) = \sum_{y\in X} g(x,y)U(y).$$

The function W is nonnegative and solves $S\,W = -U$. A computation shows that for $x \in Y$,

$$\begin{aligned} (\tilde{S}\,W)(x) &= (S\,W)(x) - \sum_{y\in E} a(x,y)[W(y) - W(x)] \\ &\le -U(x) + \sum_{y\in E} a(x,y)W(x) \\ &= -U(x) + A(x)W(x) \le -U(x) + CA(x) \end{aligned}$$

where $C = \sup_{x\in Y} W(x)$. Then

$$\begin{aligned} \sum_{y\in Y} \tilde{g}(x,y)U(y) &\le W(x) + C \sum_{y\in Y} \tilde{g}(x,y)A(y) \\ &\le W(x) + \frac{C\theta(x)}{(1-\beta)}. \end{aligned} \tag{3.8}$$

Taking the supremum over x, we get

$$\sup_{x\in Y}\sum_{y\in Y}\tilde{g}(x,y)U(y)\leq C+\frac{C\beta}{(1-\beta)}=\frac{C}{(1-\beta)},$$

which proves (3.5). Taking $U(x)$ to be $\delta_y(x)$, we have $W(x)=g(x,y)$ and $\sup_x W(x)=g(y,y)$. Hence (3.6) follows from (3.8). Taking $x=y$, we get (3.7). $\bigotimes$

Remark. If $a(x,y)$ is local then we can do better if we define $\bar{E}=\cup_{x\in E}\{y:a(x,y)>0\}$ and take $C=\sup_{x\in\bar{E}}W(x)$.

We can combine Lemmas 3.1 and 3.2 to obtain the next lemma.

Lemma 3.4 . *For any function u on Y*

$$\sum_{x\in Y}A(x)u^2(x)\leq\frac{\beta}{(1-\beta)}\tilde{I\!\!D}\,(u).$$

Proof: According to Lemma 3.1

$$\sum_{x\in Y}A(x)u^2(x)\leq C\tilde{I\!\!D}\,(u)$$

where the constant can be taken to be

$$C=\sup_{x\in Y}\sum_{y\in Y}\tilde{g}(x,y)A(y).$$

In Lemma 3.2 we established in that for $x\in Y$

$$\sum_{y\in Y}\tilde{g}(x,y)A(y)\leq\frac{\theta(x)}{(1-\beta)}.$$

Taking the supremum over x, clearly we can take C to be $\frac{\beta}{(1-\beta)}$. $\bigotimes$

The following results concerning two random walks will be useful.

Lemma 3.5 . *For the symmetric random walk on $\{Z^d\backslash E\}\times\{Z^d\backslash E\}$ with generator*

$$\sum_z a(z)[f(x_1+z,x_2)-f(x_1,x_2)]+\sum_z a(z)[f(x_1,x_2+z)-f(x_1,x_2)]$$

the probability $\theta(x_1,x_2)$ of hitting the excluded set $E_2=\{x_1=0\}\cup\{x_2=0\}\cup\{x_1=x_2\}$ has the property

$$\sup_{x_1,x_2\in(Z^d\times Z^d)\backslash E_2}\theta(x_1,x_2):=\alpha<1.$$

Proof: If we denote by $\delta(x)$ the probability that the continuous time random walk in Z^d hits 0 for some $t \geq 0$, then $\theta(x_1, x_2))$ can be bounded by

$$\theta(x_1, x_2) \leq \delta(x_1) + \delta(x_2) + \delta(x_1 - x_2).$$

We know that $\sup_{x \neq 0} \delta(x) < 1$ and $\delta(x) \to 0$ as $x \to \infty$. This is enough to conclude that $\theta(x_1, x_2)$ stays away from 1 near ∞ and since it is strictly less than 1 for each (x_1, x_2) we are done. $\otimes$

The number α is a constant of the underlying random walk and $0 < \alpha < 1$. The same α is also a bound for the hitting probability of 0 for a single random walk on Z^d. From Lemma 3.3 we get for the random walk on Z^d that excludes the origin the following bound for the Green's function

$$g_0(x, x) \leq \frac{1}{(1-\alpha)} g(x, x) = \frac{g(0, 0)}{(1-\alpha)}.$$

We now consider the symmetric simple exclusion processes on $\mathbb{Z}^d \setminus \{0\}$. The state space Ω consists of maps η from the countable set $\mathbb{Z}^d \setminus \{0\}$ into $\{0, 1\}$. Fix $0 < \rho < 1$ and denote the Bernoulli product mesure with density ρ by P_ρ. For $x \in \mathbb{Z}^d$, we define

$$\xi_x(\eta) = \frac{\eta(x) - \rho}{\sqrt{\rho(1-\rho)}}. \tag{3.9}$$

For $A \subset \mathbb{Z}^d \setminus \{0\}$ we define

$$\begin{cases} \xi_A(\eta) = \prod_{x \in A} \xi_x(\eta) \text{ if } A \text{ is nonempty} \\ \xi_A(\eta) = 1 \text{ if } A \text{ is empty.} \end{cases} \tag{3.10}$$

Then $\{\xi_A(\cdot)\}$ is an orthonormal basis for $\mathbf{H} = L_2(\Omega)$. It comes naturally graded as $\mathbf{H} = \oplus_{n \geq 0} H_n$ where H_n is the span of ξ_A over sets $\{A\}$of cardinality n.

If we write

$$u = \sum_A \tilde{u}(A) \xi_A$$

then the Dirichlet form can be calculated explicitly and we obtain

$$D^{ex}(u) = \frac{1}{2} \sum_{n \geq 1} \sum_{|A| = n} \sum_{x, y} a(x, y) [\tilde{u}(A^{x,y}) - \tilde{u}(A)]^2 := D^{ex}(\tilde{u})$$

where

$$\begin{cases} A^{x,y} = A \text{ if either } x, y \in A \text{ or } x, y \notin A \\ A^{x,y} = (A \backslash x) \cup y \text{ if } x \in A \text{ and } y \notin A \\ A^{x,y} = (A \backslash y) \cup x \text{ if } x \notin A \text{ and } y \in A. \end{cases}$$

The computation depends on the simple observation that $\xi_A(\eta^{x,y}) = \xi_{A^{x,y}}(\eta)$ for all x, y in X.

Let us denote by $\mathcal{X}$ the space of all finite susets of $\mathbb{Z}^d \setminus \{0\}$ and we write $\mathcal{X}$ as the natural union $\cup \mathcal{X}_n$ of spaces of subsets of cardinality n. We will consider functions on $\mathcal{X}$ or $\mathcal{X}_n$ and we assume initially that they are zero outside a finite set of points in $\mathcal{X}$ or $\mathcal{X}_n$. Then $\tilde{u}$ is a function on $\mathcal{X}$, and we can write it as $\tilde{u} = \sum_n \tilde{u}_n$ with $\tilde{u}_n$ being the restriction of $\tilde{u}$ to $\mathcal{X}_n$. Clearly,

$$D^{ex}(\tilde{u}) = \sum_{n \geq 1} D_n^{ex}(\tilde{u}_n).$$

where for $f : \mathcal{X}_n \to R$

$$\begin{aligned} D_n^{ex}(f) &= \frac{1}{2} \sum_{A \in \mathcal{E}_n} \sum_{x,y} a(x,y)[f(A^{x,y}) - f(A)]^2 \\ &= \sum_{A \in \mathcal{E}_n} \sum_{\substack{x \in A \\ y \notin A}} a(x,y)[f((A \backslash x) \cup y) - f(A)]^2. \end{aligned}$$

The Dirichlet form D^{sh} associated with the shift operator L^{sh} is not graded naturally according to our decomposition. A related Dirichlet form allowing the tagged particle to jump to an occupied site,

$$\hat{D}^{sh}(u) = \frac{1}{2} \sum_x a(x)[u(\tau_x \eta) - u(\eta)]^2 dP_\rho, \tag{3.11}$$

is naturally graded with

$$\hat{D}_{sh,n}(u) = \hat{D}_{sh,n}(\tilde{u}_n) = \frac{1}{2} \sum_{x,|A|=n} a(x)[\tilde{u}_n(\tau_x A) - \tilde{u}_n(A)]^2.$$

We shall prove later on an estimate of $\hat{D}^{sh}$ in terms of D^{sh} and D^{ex}.

We need the following concept later on. Suppoe $n > m$ and we are given a function g on $\mathcal{X}_n$. The function f on $\mathcal{X}_m$ is defined by $f \geq 0$ and,

$$f^2(B) = \sum_{\substack{A: A \supset B \\ |A|=n}} g^2(A).$$

Notice that if we consider g as a wave function then f^2 is simply the m point function associate with the wave function. A very important relation between f and g is the connection between their kinetic energy, especially the special case $m = 1$, which leads to the semiclassical limit for the kinetic energy of Bose gases. Notice that in our setting there is a interaction between particles, namely, the exclusion rule. Nevertheless, it is still correct.

Lemma 3.6 . *For any $n > m$*

$$D^{ex}_m(f) \leq \binom{n-1}{m-1} D^{ex}_n(g).$$

In particular, for $m = 1$ we have for the function

$$f^2(x) = \sum_{A: A \ni x} g^2(A),$$

that

$$D^{ex}_1(f) \leq D^{ex}_n(g).$$

Proof: If B_1 and B_2 are two subsets of size m that differ by one point, namely $B_2 = B_1 \backslash x \cup y$, then one has the following obvious estimate

$$|f(B_1) - f(B_2)|^2 \leq \sum_{\substack{A: |A|=n \\ x \in A, y \notin A}} |g(A \backslash x \cup y) - g(A)|^2.$$

One multiplies this inequality by $a(x, y)$ and sums over everything in sight. One has to be careful and count the number of times a term like $|g(A \backslash x \cup y) - g(A)|^2$ occurs in the summation on the right hand side. That number is clearly the number of different subsets of A of size m that include a given $x \in A$ and this yields the combinatorial prefactor $\binom{n-1}{m-1}$.

$\bigotimes$

The following lemma is the main result of this section.

Lemma 3.7 . *Let $n \geq 2$. For any set A in $\mathcal{E}_n$, i.e, a subset of $Z^d \backslash \{0\}$ of cardinality n, that consists of n distinct points $x_1, x_2, \cdots, x_n$, we define*

$$W_1(A) = \sum_i a(x_i)$$

and

$$W_2(A) = \sum_{i \neq j} a(x_i - x_j).$$

Then for any u defined on $\mathcal{E}_n$ which is in the Dirichlet Space

$$\sum_A W_1(A) u^2(A) \leq \frac{\alpha}{(1-\alpha)} \; D_n^{ex}(u)$$

and

$$\sum_A [W_1(A) + W_2(A)] u^2(A) \leq \frac{\alpha}{(1-\alpha)} \; n \; D_n^{ex}(u).$$

Proof: Define v on $\mathcal{X}_1 = Z^d \backslash 0$ by

$$v^2(x) = \sum_{A : A \ni x} u^2(A).$$

By Lemma 3.6

$$D_1^{ex}(v) \leq D_n^{ex}(u).$$

By definition of v,

$$\sum_{|A|=n} W_1(A) u^2(A) = \sum_{x \neq 0} a(x) v^2(x).$$

By Lemma 3.4 and the remark at the end of Lemma 3.6

$$\sum_{x \neq 0} a(x) v^2(x) \leq \frac{\alpha}{(1-\alpha)} D_1^{ex}(v) = \frac{\alpha}{(1-\alpha)} D_n^{ex}(u).$$

This proves the estimate involving W_1.

To prove the second inequality, define the function w on $\mathcal{X}_2$ by

$$w^2(B) = \sum_{A : A \supset B} u^2(A).$$

We can view w as a symmetric function on $(Z^d \times Z^d) \backslash E_2$ and compute its Dirichlet form,

$$D_2^{ex}(w) = \frac{1}{2} \sum_{\substack{x_1, x_2, y_1 \in Z^d \backslash \{0\} \\ x_1 \neq y_1}} a(y_1 - x_1) [w(y_1, x_2) - w(x_1, x_2)]^2$$
$$+ \frac{1}{2} \sum_{\substack{x_1, x_2, y_2 \in Z^d \backslash \{0\} \\ x_2 \neq y_2}} a(y_2 - x_2) [w(x_1, y_2) - w(x_1, x_2)]^2;$$

here the factor $\frac{1}{2}$ appears because each set B is counted twice. Again by Lemma 3.6

$$D_2^{ex}(w) \leq (n-1) D_n^{ex}(u). \tag{3.12}$$

Clearly,

$$\sum_A [W_1(A) + W_2(A)] u^2(A) = \sum_{\substack{x_1, x_2 \in Z^d \setminus \{0\} \\ x_1 \neq x_2}} H(x_1, x_2) w^2(x_1, x_2)$$

where

$$H(x_1, x_2) = a(x_1) + a(x_2) + 2a(x_1 - x_2)$$

is the sum of all the transition rates into the excluded set for the two random walks.

An easy calculation using Lemma 3.5 establishes

$$\sum_B H(B) w^2(B) = \frac{1}{2} \sum_{x_1 \neq x_2} H(x_1, x_2) w^2(x_1, x_2) \leq \frac{1}{2} \frac{\alpha}{(1-\alpha)} 2 D_2^{ex}(w).$$

Together with (3.12), this proves the lemma. $\otimes$

Another important Markovian evolution, referred to as Glauber dynamics, will be useful later on. We now collect some of its properties here. The Glauber dynamics has each site flipping back and forth between 0 and 1 as a two state Markov process. The flip rates from $1 \to 0$ and $0 \to 1$ are taken to be $\frac{1}{\rho}$ and $\frac{1}{(1-\rho)}$ respectively so that the generator for a single site is

$$(Gv)(0) = \frac{1}{(1-\rho)} (v(1) - v(0))$$
$$(Gv)(1) = \frac{1}{\rho} (v(0) - v(1))$$

with the corresponding Dirichlet form, $[v(0) - v(1)]^2$. For the full process on Ω we need to define the flip operator σ_x at site x

$$\begin{cases} (\sigma_x \eta)(y) = \eta(y) & \text{if } \; x \neq y \\ (\sigma_x \eta)(x) = 1 - \eta(x) & \text{for } \; x \in X. \end{cases}$$

The Glauber generator then has the Dirichlet form

$$D_g(u) = \sum_x D_{g,x}(u)$$

where

$$D_{g,x}(u) = \int_\Omega [u(\sigma_x \eta) - u(\eta)]^2 dP_\rho. \tag{3.13}$$

Our next lemma is an estimate of $D_{g,x}(u)$ in terms of $D^{ex}(u)$ in the transient case. The transience is important because in the Glauber dynamics "particles" are not conserved whereas in exclusion processes they are conserved. The only way to kill a particle in the exclusion model is to send it to ∞ and transience plays an important role in that.

Lemma 3.8 . *Assume transience. Let $u : \Omega \to R$ depend on a finite set of cordinates. Then for any $x \in X$,*

$$\int_{\eta(x)=1} [u(\sigma_x \eta) - u(\eta)]^2 dP_\rho \leq \frac{1}{(1-\rho)}\, g(x,x)\, D^{ex}(u)$$
$$\int_{\eta(x)=0} [u(\sigma_x \eta) - u(\eta)]^2 dP_\rho \leq \frac{1}{\rho}\, g(x,x)\, D^{ex}(u).$$

By adding the two inequalities,

$$D_{g,x}(u) \leq \frac{1}{\rho(1-\rho)} g(x,x) D^{ex}(u).$$

Proof: An elementary calculation shows that for a function

$$u(\eta) = \sum_A \tilde{u}(A)\xi_A(\eta)$$

$$\int_{\eta(x)=1} [u(\sigma_x \eta) - u(\eta)]^2 dP_\rho = \frac{1}{(1-\rho)} \sum_{A:A\ni x} \tilde{u}^2(A)$$
$$\int_{\eta(x)=0} [u(\sigma_x \eta) - u(\eta)]^2 dP_\rho = \frac{1}{\rho} \sum_{A:A\ni x} \tilde{u}^2(A).$$

Adding the two equalities we get

$$D_{g,x}(u) = \frac{1}{\rho(1-\rho)} \sum_{A:A\ni x} \tilde{u}^2(A).$$

If we define $\tilde{u}_n$ to be the restriction of $\tilde{u}$ to $\mathcal{X}_n$,

$$\sum_{A:A\ni x} \tilde{u}^2(A) = \sum_n v_n^2(x)$$

where

$$v_n^2(x) = \sum_{\substack{A:|A|=n \\ A\ni x}} \tilde{u}_n^2(A).$$

From lemma 3.1

$$v_n^2(x) \leq g(x,x)D_1(v_n)$$

and by lemma 3.6

$$D_1^{ex}(v_n) \leq D_n^{ex}(\tilde{u}_n).$$

The lemma follows. $\bigotimes$

As corollaries of previous lemmas, we prove Lemma 2.1.

Proof of Lemma 2.1. Clearly to get a bound of the form

$$| < f, v > | \leq C_f \sqrt{\sum_{x \in F} D_{g,x}(v)}$$

where f depends only on the coordinates x from F, is just finite dimensional matrix algebra. We can now use Lemma 3.8 to complete the proof. $\bigotimes$

Next we obtain an estimate of $\hat{D}^{sh}$. We already know that $D^{sh}(u) \leq \hat{D}^{sh}(u)$ as the $\hat{D}^{sh}$ form is stronger than D^{sh} form because it corresponds to the tagged particle exchanging places with an untagged particle which is not allowed. However we could make the particle at x disappear, let the tagged particle jump to x which is now empty and have our old disappeared particle reappear at $-x$ and accomplish our goal. This means that the $\hat{D}^{sh}$ form can be estimated in terms of the D^{sh} form and the Glauber forms at x and $-x$. More precisely

$$\begin{aligned}
&\int_{\eta(x)=1} [u(\tau_x \eta) - u(\eta)]^2 dP_\rho \\
&\leq 3 \int_{\eta(x)=1} \Big[[u(\sigma_{-x}\tau_x\sigma_x\eta) - u(\tau_x\sigma_x\eta)]^2 + [u(\tau_x\sigma_x\eta) - u(\sigma_x\eta)]^2 + u(\sigma_x\eta) - u(\eta)]^2 \Big] dP_\rho \\
&= 3\frac{\rho}{(1-\rho)} \int_{\eta(-x)=0} [u(\sigma_{-x}\eta) - u(\eta)]^2 + 3\frac{\rho}{(1-\rho)} \int_{\eta(x)=0} [u(\tau_x\eta) - u(\eta)]^2 dP_\rho \\
&\qquad\qquad + 3\int_{\eta(x)=1} [u(\sigma_x\eta) - u(\eta)]^2 dP_\rho.
\end{aligned}$$

Note that the maps σ_x are not measure preserving when $\rho \neq \frac{1}{2}$ and we therefore pick up the factor $\frac{\rho}{(1-\rho)}$. We can now use lemma 3.8 to conclude that for some constant C not depending on ρ

$$\hat{D}^{sh}(u) \leq \frac{C}{1-\rho}[D^{ex}(u) + D^{sh}(u)]. \tag{3.14}$$

4. Some Estimates on the Generator

Recall the generator $L = L^{ex} + L^{sh}$ of the environment process defined in (1.3) and (1.2). Note also that $L_2 = \mathbf{H} = \oplus_{n\geq 0} H_n$ where H_n is spanned by the orthonormal basis $\{\xi_A : |A| = n\}$ with ξ_A defined in (3.10). Notice that L is a bounded operator on each H_n and maps it into $H_{n-1} \oplus H_n \oplus H_{n+1}$. There are therefore bounded operators $B_{n,n-1} : H_n \to H_{n-1}$, $B_{n,n} : H_n \to H_n$ and $B_{n,n+1} : H_n \to H_{n+1}$ such that for $u \in H_n$

$$Lu = B_{n,n-1}u + B_{n,n}u + B_{n,n+1}u.$$

To compute these operators explicitly, we start with the action of L^{ex} on ξ_A,

$$L^{ex}\xi_A = \sum_{x,y\neq 0} p(y-x)\eta(x)(1-\eta(y))[\xi_{A^{x,y}} - \xi_A].$$

Recall $p(x) = a(x) + b(x)$, the sum of its symmetric and asymmetric parts. Hence we obtain

$$L^{ex}\xi_A = \frac{1}{2}\sum_{x,y\neq 0} a(y-x)[\xi_{A^{x,y}} - \xi_A] + \frac{1}{2}\sum_{x,y\neq 0} b(y-x)(\eta(x)-\eta(y))[\xi_{A^{x,y}} - \xi_A].$$

To proceed further, we write

$$\eta(x) - \eta(y) = \sqrt{\rho(1-\rho)}[\xi_x - \xi_y]$$

and use the rule

$${\xi_x}^2 = 1 + \frac{1-2\rho}{\sqrt{\rho(1-\rho)}}\xi_x$$

to obtain $[\eta(x) - \eta(y)][\xi_{A^{x,y}} - \xi_A]$ equals

$$\begin{cases} 2\sqrt{\rho(1-\rho)}[\xi_{A\cup y} - \xi_{A\setminus x}] + \frac{2\rho-1}{\sqrt{\rho(1-\rho)}}[\xi_A + \xi_{A\setminus x\cup y}] & \text{if} \quad x\in A, y\notin A \\ 2\sqrt{\rho(1-\rho)}[\xi_{A\setminus y} - \xi_{A\cup x}] - (2\rho-1)[\xi_A + \xi_{A\setminus y\cup x}] & \text{if} \quad x\notin A, y\in A \\ 0 \quad \text{in all other cases.} \end{cases}$$

This yields

$$
\begin{aligned}
L^{ex}\xi_A = &\frac{1}{2}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} a(y-x)[\xi_{A\backslash x\cup y}-\xi_A] + \frac{1}{2}\sum_{\substack{x,y\neq 0\\ x\notin A, y\in A}} a(y-x)[\xi_{A\backslash y\cup x}-\xi_A]\\
&+\frac{2\rho-1}{2}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)[\xi_{A\backslash x\cup y}+\xi_A] - \frac{2\rho-1}{2}\sum_{\substack{x,y\neq 0\\ x\notin A, y\in A}} b(y-x)[\xi_{A\backslash y\cup x}+\xi_A]\\
&+\sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\cup y} - \sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\notin A, y\in A}} b(y-x)\xi_{A\cup x}\\
&-\sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\backslash x} + \sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\notin A, y\in A}} b(y-x)\xi_{A\backslash y}.
\end{aligned}
\tag{4.1}
$$

We have arranged the terms in (4.1) so that the first line is a symmetric piece of $B_{n,n}$, the second an asymmetric piece again of $B_{n,n}$ and the next two lines are parts of $B_{n,n+1}$ and $B_{n,n-1}$ respectively. In fact there is a symmetry relative to the interchange of x and y and the first and second terms on each line are equal. We use this to rewrite

$$
\begin{aligned}
L^{ex}\xi_A = &\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} a(y-x)[\xi_{A\backslash x\cup y}-\xi_A]\\
&+(2\rho-1)\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)[\xi_{A\backslash x\cup y}+\xi_A]\\
&+2\sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\cup y}\\
&-2\sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\backslash x}.
\end{aligned}
\tag{4.2}
$$

Now we compute in a similar fashion the term $L^{sh}\xi_A$. Although the tagged particle cannot jump to a site where there is already a particle present, it is convenient to extend the definition of τ_x to the set $\{x : \eta(x) = 1\}$ by allowing the tagged particle to jump to the site $-x$ which now becomes the origin, and move the particle that was originally at $-x$ back to where the tagged particle was which is now x. This defines a version of τ_x with $\tau_x\xi_A = \xi_{\tau_x A}$ where

$$
\tau_x A = \begin{cases} A+x & \text{if } \quad -x\notin A\\ (A+x)\backslash 0\cup x & \text{if } \quad -x\in A. \end{cases}
$$

Note that $x \in \tau_x A$ if and only if $-x \in A$. After careful calculation we obtain

$$\begin{aligned}
L^{sh}\xi_A &= (1-\rho)\sum_{-x\notin A} a(x)[\xi_{\tau_x A} - \xi_A] + \rho\sum_{-x\in A} a(x)[\xi_{\tau_x A} - \xi_A] \\
&\quad + (1-\rho)\sum_{-x\notin A} b(x)[\xi_{\tau_x A} + \xi_A] + \rho\sum_{-x\in A} b(x)[\xi_{\tau_x A} + \xi_A] \\
&\quad + \sqrt{\rho(1-\rho)}\sum_{x\notin A} p(x)\xi_{A\cup x} - \sqrt{\rho(1-\rho)}\sum_{-x\notin A} p(x)\xi_{(A+x)\cup x} \\
&\quad + \sqrt{\rho(1-\rho)}\sum_{x\in A} p(x)\xi_{A\backslash x} - \sqrt{\rho(1-\rho)}\sum_{-x\in A} p(x)\xi_{(A+x)\backslash 0}.
\end{aligned} \tag{4.3}$$

Again the first line is the symmetric part of $B_{n,n}$, followed by its asymmetric part and then the pieces of $B_{n,n+1}$ and $B_{n,n-1}$ respectively.

Putting together L^{ex} and L^{sh}, we have

$$\begin{aligned}
B_{n,n}\xi_A = &\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} a(y-x)[\xi_{A\backslash x\cup y} - \xi_A] \\
&+ (1-\rho)\sum_{-x\notin A} a(x)[\xi_{\tau_x A} - \xi_A] + \rho\sum_{-x\in A} a(x)[\xi_{\tau_x A} - \xi_A] \\
&+ (2\rho - 1)\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)[\xi_{A\backslash x\cup y} + \xi_A] \\
&+ (1-\rho)\sum_{-x\notin A} b(x)[\xi_{\tau_x A} + \xi_A] + \rho\sum_{-x\in A} b(x)[\xi_{\tau_x A} + \xi_A]
\end{aligned} \tag{4.4}$$

$$B_{n,n+1}\xi_A = \sqrt{\rho(1-\rho)}\Big[\, 2\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\cup y} + \sum_{x\notin A} p(x)\xi_{A\cup x} - \sum_{-x\notin A} p(x)\xi_{(A+x)\cup x}\Big] \tag{4.5}$$

$$B_{n+1,n}\xi_A = \sqrt{\rho(1-\rho)}\Big[-2\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\backslash x} + \sum_{x\in A} p(x)\xi_{A\backslash x} - \sum_{-x\in A} p(x)\xi_{(A+x)\backslash 0}\Big]. \tag{4.6}$$

First we will provide some estimates on $B_{n,n+1}$ and $B_{n+1,n}$. If we separate out the odd and even parts

$$B_{n,n+1} = \sqrt{\rho(1-\rho)}[B^{odd}_{n,n+1} + B^{even}_{n,n+1}]$$

where the odd and even parts are given by

$$\begin{aligned}
B^{odd}_{n,n+1}(\xi_A) &= \Big[\, 2\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\cup y} + \sum_{x\notin A} b(x)\xi_{A\cup x} - \sum_{-x\notin A} b(x)\xi_{(A+x)\cup x}\Big] \\
B^{even}_{n,n+1}(\xi_A) &= \Big[\sum_{x\notin A} a(x)\xi_{A\cup x} - \sum_{-x\notin A} a(x)\xi_{(A+x)\cup x}\Big].
\end{aligned} \tag{4.7}$$

The even part of $B_{n+1,n}$ is the adjoint of the even part of $B_{n,n+1}$ and for the odd part an extra change of sign is involved. So if we bound the odd and even pieces of $B_{n,n+1}$ separately the dual bounds are valid for $B_{n+1,n}$. Our basic estimate is

Lemma 4.1 . *For all $n \geq 1$ and $u \in H_n$ and $v \in H_{n+1}$,*

$$|< B_{n,n+1}u, v >| \leq C\sqrt{\rho(1-\rho)}\sqrt{D_{ex,n+1}(v)}\left[\sqrt{n}\sqrt{D_{ex,n}(u)} + \sqrt{\hat{D}_{sh,n}(u)}\right]$$

$$|< B_{n+1,n}u, v >| \leq C\sqrt{\rho(1-\rho)}\sqrt{D_{ex,n}(u)}\left[\sqrt{n}\sqrt{D_{ex,n+1}(v)} + \sqrt{\hat{D}_{sh,n+1}(v)}\right]$$

where C is a constant that is independent of n and ρ.

Proof: We shall only prove the first estimate. The second one follows in a similar way or by invoking duality. Write functions u and v in H_n and H_{n+1} respectively with expansions $u = \sum_{|A|=n} \tilde{u}(A)\xi_A$ and $v = \sum_{|D|=n+1} \tilde{v}(D)\xi_D$. From (4.7), the estimate for $B^{odd}_{n,n+1}$ reduces to the estimation of

$$\Big|2\sum_{|A|=n}\sum_{x\in A, y\notin A} b(y-x)\tilde{u}(A)\tilde{v}(A\cup y)\Big| + \Big|\sum_{x\notin A} b(x)\tilde{u}(A)\tilde{v}(A\cup x) - \sum_{-x\notin A} b(x)\tilde{u}(A)\tilde{v}((A+x)\cup x))\Big|.$$

The first term is estimated by

$$\begin{aligned}
&\Big|2\sum_{|D|=n+1}\sum_{x,y\in D} b(y-x)\tilde{u}(D\backslash y)\tilde{v}(D)\Big| \\
&= \Big|\sum_{|D|=n+1}\sum_{x,y\in D} b(y-x)(\tilde{u}(D\backslash y) - \tilde{u}(D\backslash x))\tilde{v}(D)\Big| \quad \text{(by skew symmetry of } b) \\
&\leq \sum_{|D|=n+1}\sum_{x,y\in D} a(y-x)|\tilde{u}(D\backslash y) - \tilde{u}(D\backslash x)||\tilde{v}(D)| \qquad \text{(because } |b| \leq a) \\
&\leq \Big[\sum_{|D|=n+1}\sum_{x,y\in D} a(y-x)|\tilde{u}(D\backslash y) - \tilde{u}(D\backslash x)|^2\Big]^{\frac{1}{2}} \Big[\sum_{|D|=n+1}\sum_{x,y\in D} a(y-x)|\tilde{v}(D)|^2\Big]^{\frac{1}{2}}.
\end{aligned} \tag{4.8}$$

The first expression

$$\Big[\sum_{|D|=n+1}\sum_{x,y\in D} a(y-x)|\tilde{u}(D\backslash y) - \tilde{u}(D\backslash x)|^2\Big]$$

is seen to equal $2D_{ex,n}(u)$. In view of Lemma 3.7 the second expression is bounded by $C\ n\ D_{ex,n+1}(v)$. We turn to the second term. This leads to

$$
\begin{aligned}
&\Big| \sum_{\substack{x,D \\ x\in D, |D|=n+1}} b(x)[\tilde u(D\backslash x) - \tilde u((D\backslash x)-x)]\tilde v(D)\Big| \\
&\leq \Big[\sum_{\substack{x,D \\ x\in D, |D|=n+1}} a(x)[\tilde u(D\backslash x) - \tilde u((D\backslash x)-x)]^2\Big]^{\frac{1}{2}} \Big[\sum_{\substack{x,D \\ x\in D, |D|=n+1}} a(x)[\tilde v(D)]^2\Big]^{\frac{1}{2}} \qquad (4.9) \\
&= \Big[\sum_{\substack{x,A \\ x\notin A, |A|=n}} a(x)[\tilde u(A) - \tilde u(A-x)]^2\Big]^{\frac{1}{2}} \Big[\sum_{\substack{x,D \\ x\in D, |D|=n+1}} a(x)[\tilde v(D)]^2\Big]^{\frac{1}{2}}.
\end{aligned}
$$

We have used the inequality $|b(x)| \leq a(x)$ in the last step and used $A = D\backslash x$ as the summation variable. The first expression is estimated by

$$
\begin{aligned}
\sum_{\substack{x,A \\ x\notin A, |A|=n}} a(x)[\tilde u(A) - \tilde u(A-x)]^2 &= \sum_{\substack{x,A \\ x\notin A, |A|=n}} a(x)[\tilde u(\tau_x A) - \tilde u(A]^2 \\
&\leq \sum_{x,A:|A|=n} a(x)[\tilde u(\tau_x A) - \tilde u(A)]^2 = 2\hat D_{sh,n}(\tilde u)
\end{aligned}
$$

and the second by

$$
\sum_{\substack{x,D \\ x\in D, |D|=n+1}} a(x)[\tilde v(D)]^2 \leq \frac{\alpha}{(1-\alpha)} D_{ex,n+1}(\tilde v).
$$

The estimate for $B^{odd}_{n,n+1}$ is similar. So we have proved the first estimate of Lemma 4.1. The second one follows from duality. This concludes the lemma. $\otimes$

5. Some Estimates on the Resolvent.

The main theorem of this section is

Theorem 5.1 . *Let $0 < \rho < 1$ and a local function f be given. Then the solution u_λ of the resolvent equation (2.7) satisfies the following estimates: For every $k \geq 0$ there is a constant C_k independent of n and ρ such that*

$$
\sup_{\lambda > 0} \sum_n n^{2k} \left[D_{ex,n}(u_\lambda) + (1-\rho)\hat D_{sh,n}(u_\lambda) \right] \leq C_k\, C_f
$$

$$\sup_{\lambda>0} \lambda \sum_n n^{2k} \, \|u_{\lambda,n}\|_0^2 \leq C_k \, C_f.$$

Proof: We first recall standard estimates. Multiplying the equation (5.1) by u_λ and integrating

$$\lambda \|u_\lambda\|_0^2 + D^{ex}(u_\lambda) + D^{sh}(u_\lambda) =< f, u_\lambda > \leq C_f \sqrt{D^{ex}(u_\lambda)}.$$

This leads immediately to the estimates

$$\sup_{\lambda>0} \lambda \|u_\lambda\|_0^2 \leq C_f \tag{5.1}$$

$$\sup_{\lambda>0} D^{ex}(u_\lambda) \leq C_f \tag{5.2}$$

$$\sup_{\lambda>0} D^{sh}(u_\lambda) \leq C_f. \tag{5.3}$$

Because of the estimate (3.14) we also have

$$\sup_{\lambda>0} \hat{D}^{sh}(u_\lambda) \leq \frac{1}{(1-\rho)} C_f. \tag{5.4}$$

Let us define the operator T as multiplication by a scalar $t(n)$ on each H_n. The sequence $t(n)$ is assumed to be positive, increasing and eventually constant. Since L is bounded on each H_n and T is a multiple of I except on a finite number of H_n, it is easily verified that T leaves the domain of L invariant and the commutator $[T, L] = TL - LT$ is a bounded operator from $\mathbf{H} \to \mathbf{H}$. For $u = \sum_n u_n$ with $u_n \in H_n$, an explicit calculation yields

$$[TL - LT]u = \sum_n \left[(t(n+1) - t(n)) B_{n,n+1} u_n + (t(n-1) - t(n)) B_{n,n-1} u_n \right]$$

and

$$\begin{aligned} < [LT - TL]u, Tu > &= \sum_n t(n+1)(t(n+1) - t(n)) < B_{n,n+1} u_n, u_{n+1} > \\ &+ \sum_n t(n-1)(t(n-1) - t(n)) < B_{n-n-1} u_n, u_{n-1} > . \end{aligned} \tag{5.5}$$

From lemma 4.1, we have

$$\begin{aligned} &| < [LT - TL]u, Tu > | \\ &\leq C \sum_n t(n+1)|t(n+1) - t(n)| \sqrt{\rho(1-\rho)} \sqrt{D_{ex,n+1}(u)} \left[\sqrt{n} \sqrt{D_{ex,n}(u)} + \sqrt{\hat{D}_{sh,n}(u)} \right] \\ &+ \sum_n t(n-1)|t(n-1) - t(n)| \sqrt{\rho(1-\rho)} \sqrt{D_{ex,n-1}(u)} \left[\sqrt{n} \sqrt{D_{ex,n}(u)} + \sqrt{\hat{D}_{sh,n}(u)} \right]. \end{aligned}$$

Since $D_{ex,n}(u) = t(n)^{-2} D_{ex,n}(Tu)$, the last term is equal to

$$C \sum_n \Big| \frac{t(n+1)}{t(n)} - 1 \Big| \sqrt{\rho(1-\rho)} \sqrt{D_{ex,n+1}(Tu)} \Big[\sqrt{n} \sqrt{D_{ex,n}(Tu)} + \sqrt{\hat{D}_{sh,n}(Tu)} \Big]$$
$$+ \sum_n \Big| \frac{t(n-1)}{t(n)} - 1 \Big| \sqrt{\rho(1-\rho)} \sqrt{D_{ex,n-1}(Tu)} \Big[\sqrt{n} \sqrt{D_{ex,n}(Tu)} + \sqrt{\hat{D}_{sh,n}(Tu)} \Big].$$

We make the additional hypothesis that for every n, $t(n)$ satisfies

$$\frac{C(1+\sqrt{n})}{2} \left\{ \left| \frac{t(n+1)}{t(n)} - 1 \right| + \left| \frac{t(n-1)}{t(n)} - 1 \right| \right\} \le \delta \tag{5.6}$$

where δ will be chosen soon. From the factor $\sqrt{\rho(1-\rho)}$ we keep only $\sqrt{(1-\rho)}$ and use it only with $\hat{D}_{sh,n}(u)$ terms. Then

$$\begin{aligned} &| < [LT - TL]u, Tu > | \\ &\le \delta \sum_n \big[D_{ex,n-1}(Tu) + D_{ex,n}(Tu) + D_{ex,n+1}(Tu) + (1-\rho)\hat{D}_{sh,n}(Tu) \big] \\ &= \delta [3 D^{ex}(Tu) + (1-\rho)\hat{D}^{sh}(Tu)] \le \delta C D(Tu) \end{aligned} \tag{5.7}$$

where the constant C comes from (3.14). We pick δ so that $C\delta < \frac{1}{4}$. Let us remark that the estimates depend on T only through δ.

From the resolvent equation we have

$$\lambda T u_\lambda - L T u_\lambda = [T, L] u_\lambda + Tf.$$

Multiply both sides by Tu_λ and integrate. From (5.7) and Lemma 5.1,

$$\lambda \|Tu_\lambda\|_0^2 + D_{env}(Tu_\lambda) \le \frac{1}{4} D_{env}(Tu_\lambda) + C_{Tf} \sqrt{D^{ex}(u_\lambda)}.$$

Clearly this is sufficient to give the estimates:

$$\sup_\lambda \lambda \|Tu_\lambda\|_0^2 \le C_{Tf}$$
$$\sup_\lambda \sum_n [t(n)]^2 [D_{ex,n}(u_\lambda) + (1-\rho)\hat{D}_{sh,n}(u)] \le C_{Tf}.$$

All we need to do now is to make the choice of $t(n) = n^k$. The sequence n^k will work for large n and it is easy to fix it up for the first few values of n so that (5.8) is satisfied for the given value of δ . We then truncate it to make it level off, obtain a uniform estimate independent of the truncation level and remove the truncation. Since f is local C_{Tf} is easily controlled. $\bigotimes$.

6. Proof of Theorem 2.3.

We first rewrite $B_{n.n}$ as follows.

Lemma 6.1 . *The operator $B_{n,n}$ from $H_n \to H_n$ can be rewritten as*

$$B_{n,n}\xi_A = \sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} q(y-x)[\xi_{A\backslash x\cup y} - \xi_A] + (1-\rho)\sum_{-x\notin A} p(x)[\xi_{\tau_x A} - \xi_A] + \rho \sum_{-x\in A} p(x)[\xi_{\tau_x A} - \xi_A]$$

where $q(x) = q_\rho(x) = a(x) + (2\rho-1)b(x) = \rho p(x) + (1-\rho)p(-x)$.

Proof. We have to compare with the earlier expression and make sure that the difference is zero. For this we need to show that

$$(2\rho-1)\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x) + (1-\rho)\sum_{-x\notin A} b(x) + \rho\sum_{-x\in A} b(x) = 0.$$

If we use the antisymmetry of $b(\cdot)$ as well as its consequence $\sum_x b(x) = 0$, the above relation is easily seen to be true. $\otimes$

Our next step is to make $B_{n,n}$ look more like a convolution operator. The full translation symmetry is not available because the set $\mathcal{X}_n$ corresponds only to distinct nonzero $n-$tuples. A subset $A \subset Z^d\backslash 0$ of cardinality n is really an equivalence class of $n!$ points in $(Z^d)^n$. All the functions that we consider on $(Z^d)^n$ will be symmetric under permutation. Let us denote by $\mathcal{G}_n \subset (Z^d)^n$ the collection of distinct ordered nonzero n-tuples. A function on $\mathcal{E}_n$ can be considered as a symmetric function on $\mathcal{G}_n$ and then extended to all of $(Z^d)^n$ by defining it to be 0 on the complement $(Z^d)^n\backslash\mathcal{G}_n$. We will decompose this complement into three parts.

$$\mathcal{B}_n^1 = \{(x_1,\cdots,x_n) : x_i \neq 0 \ \text{ for } \ 1\le i\le n \text{ and } \ x_i = x_j \ \text{ for exactly one pair } \}$$

$$\mathcal{B}_n^2 = \{(x_1,\cdots,x_n) : x_i \neq x_j \ \text{ for } \ 1\le i\neq j\le n \ \text{ and for exactly one } \ i, \ x_i = 0\}$$

and

$$\mathcal{B}_n^3 = \mathcal{G}_n^c\backslash(\mathcal{B}_n^1\cup\mathcal{B}_n^2).$$

We want to replace the operator $B_{n,n}$ acting on functions defined on $\mathcal{G}_n$ by the following operator $C_{n,n}$ of convolution type acting on the space of functions defined on all of $(Z^d)^n$. Note that $\mathcal{B}_n^1$ and $\mathcal{B}_n^2$ are the boundary of $\mathcal{G}_n$ and transitions are possible from $\mathcal{G}_n$ only into $(\mathcal{G}_n\cup\mathcal{B}_n^1\cup\mathcal{B}_n^2)$.

$$(C_{n,n}v)(x_1,\cdots,x_n) = \sum_{x,j} q(x)\ [v(x_1,\cdots,x_j+x,\cdots,x_n) - v(x_1,\cdots,x_n)]$$
$$+ (1-\rho)\sum_x p(x)\ [v(x_1-x,\cdots,x_j-x,\cdots,x_n-x) - v(x_1,\cdots,x_n)].$$

A comparison has to be made with $B_{n,n}$.

$$\begin{aligned}(B_{n,n}u)(x_1,\cdots,x_n) = &\sum_j \sum_{(x_1,\cdots,y,\cdots,x_n)\in\mathcal{G}_n} q(y-x_j)\ [u(x_1,\cdots,y,\cdots,x_n) - u(x_1,\cdots,x_n)]\\ &+ (1-\rho)\sum_{x\notin\{x_1,\cdots,x_n\}} p(x)\ [u(x_1-x,\cdots,x_n-x) - u(x_1,\cdots,x_n)]\\ &+ \rho\sum_j p(x_j)[u(x_1-x_j,\cdots,-x_j,\cdots,x_n-x_j) - u(x_1,\cdots,x_n)].\end{aligned}$$

Given a function u on $\mathcal{X}_n$ we can view it as a symmetric function defined on $\mathcal{G}_n$ and extend it to all of $(Z^d)^n$ as a symmetric function by making it 0 outside $\mathcal{G}_n$. We will abuse the notation somewhat and not distinguish between the three versions of the same function on $\mathcal{X}_n$, on $\mathcal{G}_n$ and on $(Z^d)^n$. Since we will only deal with symmetric functions it will not matter. We can also extend in the same fashion the function $f = B_{n,n}u$. We define h by

$$C_{n.n}u = B_{n.n}u + h$$

and try to estimate h in terms of u. First let us compute h explicitly. On $\mathcal{B}_n^3$ the function h is identically zero. For a point in $\mathcal{B}_n^2$, consisting of n distinct points $x_1,\cdots,x_n$ exactly one of which is 0,

$$\begin{aligned}&h(x_1,\cdots,x_{n-1},0)\\ &= \sum_{x_n:(x_1,\cdots,x_n)\in\mathcal{G}_n} q(x_n)u(x_1,\cdots,x_{n-1},x_n) + (1-\rho)\sum_{x\neq x_1,\cdots,x_n} p(x)u(x_1-x,\cdots,x_n-x).\end{aligned}$$

On $\mathcal{B}_n^1$ where a typical point is $(x_1,\cdots,x_{n-1},x_{n-1})$ with distinct nonzero $x_1,\cdots,x_{n-1}$,

$$h(x_1,\cdots,x_{n-1},x_{n-1}) = 2\sum_{x_n:(x_1,\cdots,x_n)\in\mathcal{G}_n} q(x_n-x_{n-1})u(x_1,\cdots,x_{n-1},x_n)$$

Finally on $\mathcal{G}_n$,

$$\begin{aligned}&h(x_1,\cdots,x_{n-1},x_n)\\ &= -\left[\sum_{i\neq j} q(x_i-x_j) + \sum_i q(-x_i)\right] u(x_1,\cdots,x_n) - (1-\rho)\left[\sum_j p(x_j)\right] u(x_1,\cdots,x_n)\\ &\quad -\rho\sum_j p(x_j)\,[\ u(x_1-x_j,\cdots,-x_j,\cdots,x_n-x_j) - u(x_1,\cdots,x_n)\ ]\,.\end{aligned}$$

There are two new Dirichlet forms. The forms $D_{ex,n}(\cdot)$ that we already saw on functions defined on $\mathcal{X}_n$ as well as the new Dirichlet forms:

$$\bar{D}_n(u) = \frac{1}{2}\sum_i \sum_{x_1,\cdots,x_i,x_i'\cdots,x_n} a(x_i - x_i')\,[u(x_1,\cdots,x_i,\cdots x_n) - u(x_1,\cdots,x_i',\cdots x_n)]^2$$

that corresponds to n free random walks and

$$\bar{D}_{sh,n}(u) = \frac{1}{2}\sum_{x,x_1,\cdots,x_n} a(x)[u(x_1 - x,\cdots,x_n - x) - u(x_1,\cdots,x_n)]^2$$

that corresponds to shifts. We have the following estimates.

Lemma 6.3. *If u is any symmetric function of $x_1,\cdots,x_n$ on $(Z^d)^n$*

$$D_{ex,n}(u) \le \frac{1}{n!}\bar{D}_n(u).$$

If in addition $u \equiv 0$ outside $\mathcal{G}_n$, then for some constant C independent of n and u,

$$\frac{1}{n!}\bar{D}_n(u) \le C\,D_{ex,n}(u) \quad \text{and}$$

$$|\frac{1}{n!}\bar{D}_{sh,n}(u) - \hat{D}_{sh,n}(u)| \le C D_{ex,n}(u).$$

Proof. The first part is obvious. The factorial is just the number of times each term is counted. Clearly

$$\bar{D}_n(u) = n! D_{ex,n}(u) + 2(n-1)! \sum_{x_1,\cdots,x_n} a(x_n - x_{n-1})\; u^2(x_1,\cdots,x_n)$$

and by Lemma 3.7

$$\sum_{x_1,\cdots,x_n}\sum_{i\ne j} a(x_i - x_j)u^2(x_1,\cdots,x_n) \le C\,n\,D_{ex,n}(u)$$

and now the second part follows. The third part also follows from Lemma 3.7 and is nearly identical:

$$\Big|\frac{1}{n!}\bar{D}_{sh,n}(u) - \hat{D}_{sh,n}(u)\Big| \le C\sum_{A\in\mathcal{E}_n}\big(\sum_{x\in A} a(x)\big)[u(A)]^2$$

where we have identified A with $(x_1,\cdots,x_n)$. $\otimes$

We now return to the estimation of h. If we define

$$< h,w >= \frac{1}{n!}\sum h(x_1,\cdots,x_n)\,(w(x_1,\cdots,x_n),$$

then $| < h,w > |$ is less than

$$\begin{aligned}&\le C\Big[\frac{1}{n!}\sum_{x_1,\cdots .x_n} A(x_1,\cdots .x_n)u^2(x_1,\cdots .x_n)\Big]^{\frac{1}{2}}\Big[\frac{1}{n!}\sum_{x_1,\cdots .x_n} A(x_1,\cdots .x_n)w^2(x_1,\cdots .x_n)\Big]^{\frac{1}{2}}\\ &\le C\,n\,[D_{ex,n}(u)]^{\frac{1}{2}}\big[\frac{1}{n!}\bar{D}(w)\big]^{\frac{1}{2}}.\end{aligned} \tag{6.1}$$

We have defined

$$A(x_1,\cdots,x_n) = \sum_{i\neq j} a(x_i - x_j) + \sum_j a(x_j).$$

We have used the following facts: the symmetric part q is also a; p,q are dominated by $2a$.

The next lemma is simple consequence of Fourier analysis.

Lemma 6.4. *Let u be a symmetric function of n variables satisfying*

$$\lambda u - C_{n,n}u = v$$

where v satisfies the bound

$$\Big|\sum_{x_1,\cdots,x_n} v(x_1,\cdots,x_n)\,w(x_1,\cdots,x_n)\Big| \le C\big[\bar{D}_n(w)\big]^{\frac{1}{2}}.$$

Then λu satisfies the same bound with the same constant,

$$\Big|\sum_{x_1,\cdots,x_n} \lambda u(x_1,\cdots,x_n)\,w(x_1,\cdots,x_n)\Big| \le C[\bar{D}_n(w)]^{\frac{1}{2}}.$$

Proof: Denoting by $\mathbf{T}^n$ the n-torus and by $\hat{u}(\theta)$, $\hat{v}(\theta)$ the Fourier transforms of u and v respectively, we have

$$[\lambda + \Phi(\theta) + i\Psi(\theta)]\hat{u}(\theta) = \hat{v}(\theta)$$

where $\Phi(\theta) + i\Psi(\theta)$ equals

$$\sum_j\sum_x q(x)[1-\cos(\theta_j x) - i\ \sin(\theta_j x)] + (1-\rho)\sum_x p(x)\big[1-\cos(x\sum_j\theta_j) + i\sin(x\sum_j\theta_j)\big].$$

Φ and Ψ are real and $\Phi(\theta) \geq 0$. Moreover

$$\|v\|_{-1}^2 = \int_{\mathbf{T}^n} \frac{|\hat{v}(\theta)|^2}{H(\theta)} d\theta$$

where

$$H(\theta) = \sum_j \sum_x a(x)[1 - \cos(\theta_j x)]$$

and

$$\|\lambda u\|_{-1}^2 = \int_{\mathbf{T}^n} \frac{|\lambda \hat{U}(\theta)|^2}{H(\theta)} d\theta = \int_{\mathbf{T}^n} \frac{|\hat{v}(\theta)|^2}{H(\theta)} \frac{|\lambda|^2}{|\lambda + \Phi(\theta) + i\Psi(\theta)|^2} d\theta \leq \int_{\mathbf{T}^n} \frac{|\hat{v}(\theta)|^2}{H(\theta)} d\theta$$

which proves the lemma. $\bigotimes$

We are now ready to prove our main theorem. We start with a solution u_λ of (6.1). with a local f. If we decompose and write $u = \sum_n u_n$ and $f = \sum_n f_n$ with u_n and f_n from H_n, following the notation of (4.5) and (4.6)

$$\lambda u_{\lambda,n} - B_{n.n} u_{\lambda,n} = f_n + B_{n-1,n} u_{\lambda,n-1} + B_{n+1,n} u_{\lambda,n+1} = g_n.$$

As before we rewrite

$$\lambda u_{\lambda,n} - C_{n.n} u_{\lambda,n} = g_n + h_n.$$

From (6.1) and Theorem 5.1 we have the estimate

$$| < h_n, w > | \leq C_f \, C_k \, n^{-k} \, [\frac{1}{n!} \bar{D}(w)]^{\frac{1}{2}}.$$

From lemmas 4.1 and 6.3 and Theorem 5.1 we have

$$| < w, B_{n-1,n} u_{\lambda,n-1} + B_{n+1,n} u_{\lambda,n+1} > | \leq C_f \, C_k \, n^{-k} \, [\frac{1}{n!} \bar{D}(w)]^{\frac{1}{2}}.$$

If f is local, $f_n = 0$ for large enough n. By Lemma 6.4, we now conclude that for all large n,

$$| < \lambda u_{\lambda,n}, w > | \leq C_f \, C_k \; n^{-k} \, [\frac{1}{n!} \bar{D}(w)]^{\frac{1}{2}}.$$

For small n where $f_n \neq 0$, we have from the proof of Lemma 2.1 and Lemma 6.3 that

$$| < w, f_n > | \leq C_f \sqrt{D_{ex,n}} \leq C_f \, [\frac{1}{n!} \bar{D}(w)]^{\frac{1}{2}}$$

and so for small n by Lemma 6.4,

$$|<\lambda u_{\lambda,n}, w>| \leq C_f \,\big[\frac{1}{n!}\bar{D}(w)\big]^{\frac{1}{2}}.$$

Finally by the use of Lemma 6.3, adding up contributions with a larger constant,

$$\|\lambda u_\lambda\|_{-1} \leq C_f \, C_k \sum_n n^{-k}$$

and we are done.

7. Proof of Theorem 2.7.

We now start on the lower bounds. The martingale $\xi_1(t)$ has the representation in terms of

$$\xi_1(t) = \sum_x \int_0^t c_x(s) dM_x(s)$$

with the variance

$$E\big\{ <\xi_1(1), a>^2 \big\} = (1-\rho)\sum_x p(x) <x, a>^2,$$

and the martingale $\xi_2(t)$ has a similar representation

$$\xi_2(t) = \int_0^t d_x(s) dM_x(s) + \int_0^t d_{x,y} dM_{x,y}(s)$$

with the variance

$$E\big\{ <\xi_2(1), a>^2 \big\} = D^{sh}(w) + D^{ex}(w),$$

where w comes from Theorem 2.4, and the separate terms above are variances from each corresponding martingale group. The martingales are mutually orthogonal and cannot cancel each other. For total cancellation to occur $D^{sh}(w)$ has to equal $(1-\rho)\sum_x p(x) < x, a >^2$ and $D^{ex}(w)$ has to be zero. The clue to a lower bound is to estimate $D^{sh}(w)$ in terms of $D^{ex}(w)$ so that the latter cannot go to 0 without the former going to 0 at the same time. We now proceed towards that goal.

Lemma 7.1 . *For any n complex numbers $z_1, \cdots, z_n$ of modulus 1*

$$|1 - z_1 z_2 \cdots z_n| \leq \sum_i |1 - z_i|.$$

In particular taking $z_j = \exp[\, i\, \theta_j]$,

$$[1 - \cos(\sum_{j=1}^n \theta_j)] \leq n \sum_{j=1}^n (1 - \cos\theta_j).$$

Proof: The first part is by induction and the second part by computation. $\otimes$

Lemma 7.2 . *On $(Z^d)^n$ we have the inequaity*

$$\bar{D}_{sh,n}(f) \leq n\bar{D}_n(f).$$

Proof: Compute using Fourier transform and use Lemma 7.1. $\otimes$

Lemma 7.3 .

$$D^{sh}(u) \leq C \sum_n nD_{ex,n}(u) \leq C[D^{ex}(u)]^{\frac{1}{2}} \|f\|_{-1}.$$

Proof: Recall the discussion near (3.11) and (3.14). We have

$$\begin{aligned}
D^{sh}(u) &\leq \hat{D}^{sh}(u) \\
&= \sum_n \hat{D}_{sh,n}(u) \\
&\leq \sum_n \frac{1}{n!} \bar{D}_{sh,n}(u) + C \sum_n D_{ex,n}(u) \\
&\leq \sum_n \frac{n}{n!} \bar{D}_n(u) + C \sum_n D_{ex,n}(u) \\
&\leq C \sum_n n\, D_{ex,n}(u) \\
&\leq C \, [\sum_n D_{ex,n}(u)]^{\frac{1}{2}} \, [\sum_n n^2 D_{ex,n}(u)]^{\frac{1}{2}} \\
&\leq C \, [D^{ex}(u)]^{\frac{1}{2}} \, \|Tf\|_{-1} \\
&\leq C \, [D^{ex}(u)]^{\frac{1}{2}} \, \|f\|_{-1}
\end{aligned}$$

where the third line comes from Lemma 6.3, the fourth from Lemma 7.2, the fifth from Lemma 6.3, the sixth from Schwarz inequality, the seventh from Theorem 5.1, and the last line from the local nature of f. $\otimes$

Lemma 8.7.

$$D^{sh}(u) \leq C\sqrt{1-\rho} \; [D^{ex}(u)]^{\frac{1}{2}}.$$

Proof:

$$\begin{aligned}
&E^{P_\rho}\{\sum_x p(x) < x,a > (\eta(x)-\rho))U(\eta)\}\\
&\quad = \sum_x p(x) < x,a > E^{P_\rho}\{(\eta(x)-\rho))U(\eta)\}\\
&\quad \leq \Big[\sum_x p(x)| < x,a > |^2\Big]^{\frac{1}{2}}\Big[\sum_x p(x)\big[E\{(\eta(x)-\rho))U(\eta)\}\big]^2\Big]^{\frac{1}{2}}\\
&\quad \leq C\rho(1-\rho)\Big[\sum_x p(x)\,| < x,a > |^2\Big]^{\frac{1}{2}}\big[\sum_x p(x)D_{g,x}(U)\big]^{\frac{1}{2}}\\
&\quad \leq C\sqrt{1-\rho}\,[D^{ex}(U)]^{\frac{1}{2}}
\end{aligned}$$

where the penultimate line follows from direct computation and the last line from Lemma 3.8. Therefore $\|f\|_{-1} \leq C\sqrt{1-\rho}$ and the proof is complete. $\bigotimes$

Geometrically the problem is the following:

We have three vectors U, V, W in a Hilbertspace $H = H_1 \oplus H_2$ with $U, V \in H_1$ and $W \in H_2$. We have a lower bound

$$\|U\|^2 \geq C_1(1-\rho)$$

and an upper bound

$$\|V\|^2 \leq C_2\sqrt{1-\rho}\|W\|$$

We need a lower bound on $\|U+V+W\|$. The homogeneity is just right. If we let

$$U' = (1-\rho)^{-\frac{1}{2}}U, \quad V' = (1-\rho)^{-\frac{1}{2}}V, \quad \text{and} \quad W' = (1-\rho)^{-\frac{1}{2}}W$$

then the bounds become $\|U'\| \geq C_1$ and $\|V'\|^2 \leq C_2\|W'\|$. With that we have,

$$\begin{aligned}
\|U'+V'+W'\| &\geq \Big|\|U'\|-\|V'\|\Big| + \frac{1}{C_2}\|V'\|^2\\
&\geq \inf_{y\geq 0}\Big[\ \ |C_1-y| + \frac{y^2}{C_2}\ \ \Big]\\
&= C_3 > 0
\end{aligned}$$

Therefore we have proved

Theorem 8.8.

$$< \mathbf{C}a, a > \geq C_3(1-\rho) < a, a > .$$

$\bigotimes$

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