Spin Depolarization Decay Rates in α-Symmetric Stable Fields on Cubic Lattices [1] [2]

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Abstract

We study the asymptotic long time behavior of the energy function

$$E(t;\lambda;f) = \frac{1}{t}\ln E\exp\{-tf[\frac{1}{t}\sum_{x\in\mathbf{Z^d}}(\lambda\int_0^t \delta_{\{x\}}(X_s)\,ds)^\alpha]\}$$

where $\{X_s : 0 \le s < \infty\}$ is the standard random walk on the d-dimensional lattice $\mathbf{Z^d}$; $1 < \alpha \le 2$ and $f : \mathbf{R}^+ \to \mathbf{R}^+$ is any nondecreasing concave function. In the special case $f(x) = x$, our setting represents a lattice model for the study of transverse magnetization of spins diffusing in a homogeneous α-stable i.i.d random longtitudinal field $\{\lambda V(x) : x \in \mathbf{Z^d}\}$ with common marginal, the standard α-symmetric stable distribution; the parameter λ describes the intensity of the field.

Using large deviation techniques, we show that $S_c(\lambda,\alpha,f) = \lim_{t\to\infty} E(t;\lambda;f)$ exists. Moreover, we obtain a variational formula for this decay rate S_c. Finally we analyze the behavior $S_c(\lambda,\alpha,f)$ as $\lambda \to 0$ when $f(x) = x^\beta$ for all $1 \ge \beta > 0$.

Consequently, several physical conjectures with respect to lattice models of transverse magnetization are resolved by setting $\beta = 1$ in our results. We show that $S_c(\lambda,\alpha,1) \sim \lambda^\alpha$ for $d \ge 3$, $\lambda^\alpha(\ln\frac{1}{\lambda})^{\alpha-1}$ in $d = 2$ and $\lambda^{\frac{2\alpha}{\alpha+1}}$ in $d = 1$.

1 The Model and Main Results

In [MD1, Do], an analysis of the tranverse magnetization of spins diffusing in a random longtitudinal field is proposed. They give convincing physical arguments when the randomness is Gaussian or that from an α-stable symmetric distribution. In [CX], these arguments are substantiated using large deviation techniques. Mitra and Doussal in [MD1] as well address the question of universality by developing a discrete lattice model where similar results to the continuum case are derived.

We now describe this model: Let $\{V(x) : x \in \mathbf{Z^d}\}$ be a i.i.d random field with common distribution the α-stable symmetric distribution whose characteristic function is

$$< \exp\{itV(x)\} >= \exp\{-|t|^\alpha\},$$

where $< \cdot >$ is expectation with respect to the α-random field $\{V(x) : x \in \mathbf{Z^d}\}$. Consider the following evolution equation governing the "magnetization of spins", $u(t,x)$, diffusing in the α-random field:

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2} \triangle u(t,x) + i\lambda V u(t,x), \qquad u(0,x) = 1, \tag{1.1}$$

where $\triangle$ is the usual discrete Laplace operator on $\mathbf{Z^d}$ and $i = \sqrt{-1}$. The quantity of interest, the "tranverse magnetization" is given by the expectation

$$M(t;\lambda) =< u(t,x) >$$

independent of x due to the stationarity of the α-random field. An immediate consequence of the Feynmann-Kac formula gives that

$$M(t;\lambda) = \exp\{- \sum_{x\in\mathbf{Z^d}} (\lambda \int_0^t \delta_x(X_s)\, ds)^\alpha\}, \tag{1.2}$$

where $X.$ is the standard continuous random walk on $\mathbf{Z^d}$ with generator the discrete Laplacian $1/2\triangle$. The physical results in [MD1] cite that $M(t;\lambda)$ exponentially decays in time and that the decay coefficients, $\lambda \to 0$, exhibit different behaviors depending on the lattice dimension d. We will make this more precise in a moment.

In another point of view, functionals such as $M(t;\lambda)$ in (1.2) continue to be of great interest from the perspective of random polymer models. Some work in this regard may be found in [W, K, B, GH, HH]. In the polymer context, $M(t;\lambda)$ is identified as the partition function for the Hamiltonian

$$H(t;\lambda) = \sum_{x\in\mathbf{Z^d}} (\lambda \int_0^t \delta_x(X_s)\, ds)^\alpha.$$

In this paper, we consider a Hamiltonian

$$H(t;\lambda;f) = tf(\frac{1}{t} \sum_{x\in\mathbf{Z^d}} (\lambda \int_0^t \delta_x(X_s)\, ds)^\alpha), \tag{1.3}$$

where $\lambda > 0$ and $f : \mathbf{R}^+ \to \mathbf{R}^+$ is a nondecreasing concave function (in the "magnetization" context, $f(x) = x$). We now specify throughout this article that $1 < \alpha \leq 2$.

As mentioned earlier, the "magnetization" is reasoned to experience an exponential time decay. Therefore our first intention is to study the long time behavior of the object

$$E(t;\lambda;f) = \frac{1}{t} \ln E[\exp\{-H(t;\lambda;f)\}] \tag{1.4}$$

where E is expectation with respect to the random walk. We prove below that the limit exists

$$S_c(\lambda, \alpha, f) = -\lim_{t\to\infty} E(t). \tag{1.5}$$

¿From the view of polymer and statistical physics models, this exponent is simply the pressure function whose study, as is well known, is the first step toward understanding the Gibbs states of the dynamics. Our primary aim is to investigate the behavior of $S_c(\lambda, \alpha, f)$ as $\lambda \to 0$ in different dimensions.

To prepare for the main results, we develop some notation. Consider the space $\mathbf{\Omega} = \mathbf{D}(\mathbf{R}, \mathbf{Z^d})$ of all right continuous functions $\omega : \mathbf{R} \to \mathbf{Z^d}$ with discontinuities of only the first kind and $\omega(0) = 0$. Similarly, we may define the space $\mathbf{\Omega_{a,b}} = \mathbf{D}([\mathbf{a}, \mathbf{b}], \mathbf{Z^d})$. Place on $\mathbf{\Omega}$ the standard topology induced by the Skorohod convergence on bounded intervals of the real line. Define by $\mathbf{P_{si}}(\mathbf{\Omega})$ the set of the all probability measures on $\mathbf{\Omega}$ which govern processes with stationary increments. Again, $\mathbf{P_{si}}(\mathbf{\Omega})$ will be a Polish space when endowed with the usual weak topology. Denote by E^P expectation with respect to $P \in \mathbf{P_{si}}(\mathbf{\Omega})$. Let Q_0 denote the law induced by the random walk X_s on $\mathbf{\Omega}$ (extend for instance another independent random walk in the negative time direction). Obviously $Q_0 \in \mathbf{P_{si}}(\mathbf{\Omega})$.

For any $P \in \mathbf{P_{si}}(\mathbf{\Omega})$, let $P[-t, t]$ be the restriction of P on $\mathbf{\Omega_{-t,t}}$. We may define the relative entropy of $P[-t, t]$ with respect to $Q_0[-t, t]$ by the formula

$$H_t(P|Q_0) = \sup_f \{ \int f \, dP - \ln[\int \exp(f) \, dQ_0] \},$$

where supremum is over all bounded continuous functions on $\mathbf{\Omega_{-t,t}}$. It is not hard to see that that $H_t(P|Q_0)$ is subadditive in t if we note that Q_0 is an independent stationary increament process:

$$H_{t+s}(P|Q_0) \le H_t(P|Q_0) + H_s(P|Q_0).$$

Therefore, the following definition for the relative entropy, $H(P|Q_0)$, of P with respect to Q_0 is meaningful:

$$H(P|Q_0) = \lim_{t\to\infty} \frac{1}{2t} H_t(P|Q_0). \tag{1.6}$$

Now we are at a point to state our main results:

Theorem 1.1 *Let $f : \mathbf{R}^+ \to \mathbf{R}^+$ be concave and nondecreasing. Also let $1 < \alpha \leq 2$. Then*

$$\lim_{t\to\infty} E(t;\lambda;f) = -\inf_{P\in\mathbf{P_{si}(\Omega)}} \{f(|\lambda|^\alpha E^P[\int_{-\infty}^{\infty} \delta_0(\omega(s))\,ds]^{\alpha-1}) + H(P|Q_0)\}. \qquad (1.7)$$

Remarks for Theorem 1.1 :

1. The variational formula above allows us to derive easily upper bounds for the decay rate $S_c(\lambda, \alpha, f)$. For instance we may restrict the variational problem to those laws induced by random walks with different jump rates.
2. The minimizers of this variational problem themselves are interesting objects. We believe that they are Gibbs states (as far as we know, few have been known for this problem). More interestingly, it is believed that all minimizers in low dimensions, $d \leq 3$, are long range correlated. Some interesting progress has been made in this connection [W, B, GH, HH].
3. From valuable discussions with Prof. E. Bolthausen, we have learned that similar variational formulae have been obtained in [K]. Our result, in any case, has broader scope as it treats the case of nonlinear f, seemingly beyond the methods in [K].
4. The existence of the decay rate $S_c(\lambda, \alpha, f)$ could be derived by establishing subadditivity of $E(t;\lambda;f)$ in t.

For functions of the form $f(x) = x^\beta$, set

$$S_c(\lambda, \alpha, \beta) = S_c(\lambda, \alpha, f).$$

Also, we understand the notation $g_1(\lambda) \sim g_2(\lambda)$ as $\lambda \to 0$ to mean that

$$0 < \liminf_{\lambda\to 0} \frac{g_1(\lambda)}{g_2(\lambda)} \leq \limsup_{\lambda\to 0} \frac{g_1(\lambda)}{g_2(\lambda)} < \infty.$$

Theorem 1.2 *For each $0 < \beta \leq 1$ and $d \geq 3$ we have*

$$\lim_{\lambda\to 0} \frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\alpha\beta}} = \{E^{Q_0}(\int_{-\infty}^{\infty} \delta_0[\omega(s)]\,ds)^{\alpha-1}\}^\beta < \infty. \qquad (1.8)$$

For $d = 2$ as $\lambda \to 0$ we have

$$S_c(\lambda, \alpha, \beta) \sim \lambda^{\alpha\beta}(\ln\frac{1}{\lambda})^{(\alpha-1)\beta}. \qquad (1.9)$$

For $d = 1$ as $\lambda \to 0$ we have

$$S_c(\lambda, \alpha, \beta) \sim \lambda^{\frac{2\alpha\beta}{2+\beta(\alpha-1)}}. \tag{1.10}$$

Remarks for Theorem 1.2 :

1. Our results applied to the case $\beta = 1$ affirm exactly the conjectures made in [MD1, Do].

2. In the case $\beta = 1$ and $\alpha = 2$, the Gaussian case, further limit type results in $d = 2$ follow easily from our arguments. We learned from our discussion with Bolthausen a direct argument for this limit result based on estimations of the variance of the random walk self intersection time.

As noted earlier, a similar program has been carried out recently in [CX] in the context Brownian motion and Gaussian random fields. The method developed there, based on large deviation theory and the entropy inequality, is quite flexible and we will adopt this approach to our more general context. The main problem is that by considering α-stable random fields, our Hamiltonian lacks linearity; some tricks therefore are used to circumvent the nonlinearity.

The main idea to prove Theorem 1.1 is to use a certain large deviation principle. It is not hard to see that those "paths" which contribute most in (1.4) possess tendencies to move far away with drift. These are not the typical paths for random walk; in fact exponentially small chances are attached to such paths. This gives a heuristic indication of why large deviations might play an important role in this problem.

To prove Theorem 1.2, a modified random walk measure is introduced to verify that a minimizer exists for the variational problem in Theorem 1.1. Then by entropy comparisons of this modified measure with the minimizer, the various decay rates in different dimensions are deduced.

The paper is organized as follows: In section 2, we will establish a large deviation principal for the stationary "derivatives" of random walks. In section 3, we will then prove Theorem 1.1. In section 4, we will derive that part of Theorem 1.2 dealing with $d \geq 3$. In section 5, we will complete the proof of Theorem 1.2 for the lower dimensional cases.

A survey of earlier work on magnetic depolarization, from both physical and mathematical points of view, are listed in the references of [CX, Do]. [Do] is an expository paper on this topic from a physics standpoint. Earlier work on random polymer models may be found in the references of [B, HH].

2 Large Deviation Principle for Increments of Random Walk

In this section, we will establish a large deviation principle for the "derivatives," the infinitessimal increments of standard continuous time random walk. Of course, to deal with the infinitessimals directly, we must work with temporal distributions, topologically unfriendly objects. So instead, we look their integrals, the increments, directly. In spirit however, one should view the following large deviation principle as one for "white noise". In this sense, our large deviation principle is nothing more than a direct extension of level-3 large deviations for i.i.d random sequences (see [El] for a discussion on level-3 large deviations).

Although this large deviation principle, in the way we interpret it, may be accepted easily by specialists, strictly speaking, it hasn't been appeared in the literature; as far as we know, a suitable formulation is given in [CX] for the first time. We believe that this l.d. principle is useful for other problems. For our context, we are unable to fit what we require in terms of a l.d. principle into any general framework even though large deviations theory is now quite developed.

Let, as before in section 1, $\mathbf{\Omega}$ be the space of right continuous paths $w(\cdot)$ and Q_0 the measure governing random walk on this space. $\mathbf{P_{si}}(\mathbf{\Omega})$ is again the space of measures with stationary increments on $\mathbf{\Omega}$.

For any $t > 0$ and a path $w(\cdot) \in \mathbf{\Omega}$, define the new trajectory

$$w_t(s) = \begin{cases} w(s) & \text{if } 0 \leq s \leq t \\ kw(t) + w(r) & \text{if } s = kt + r, \end{cases}$$

where k is a non-negative integer and $0 \leq r < t$. Essentially the formal derivative of $w_t(s)$ in s is the t-periodic extension of the formal derivative of $w(s)$.

Now define, for measurable $A \subset \mathbf{\Omega}$ and path ω, the empirical measure $L_t(d\omega', \omega)$:

$$L_t(A, \omega) = \frac{1}{t} \int_0^t \chi_A(\omega_t(s + \cdot) - \omega_t(s)) \, ds, \tag{2.1}$$

where χ_A is the characteristic function of A. It is not hard to see that $L_t(\cdot,\omega) \in \mathbf{P_{si}(\Omega)}$.

The following theorem, which specifies the large deviation principle for $L_t(\cdot,\omega)$ under Q_0, is the main result of this section.

Theorem 2.1 *A l. d. principle holds for the laws of $L_t(\cdot,\omega)$ on $\mathbf{\Omega}$ under $\bar{Q}_0$ as $t \to \infty$ with rate function $H(\cdot|Q_0)$, the relative entropy function w.r.t. Q_0.*

We have for closed sets $C \subset \mathbf{P_{si}(\Omega)}$

$$\limsup_{t\to\infty} \frac{1}{t} \ln Q_0\{\omega : L_t(\cdot,\omega) \in C\} \leq - \inf_{P\in C} H(P|Q_0). \tag{2.2}$$

And for open sets $O \subset \mathbf{P_{si}(\Omega)}$,

$$\liminf_{t\to\infty} \frac{1}{t} \ln Q_0\{\omega : L_t(\cdot,\omega) \in O\} \geq - \inf_{P\in O} H(P|Q_0). \tag{2.3}$$

Remarks for Theorem 2.1:

1. The proof of this result follows arguments in sec. 2 of [CX] closely except the part of exponential tightness. Therefore we only provide a proof for this part and skip the rest of arguments. One should consult [CX] for details.

2. Properties of $H(P|Q_0)$ as a rate function are established in [DV2]. We note some facts that we will use: (i) $H(P|Q_0)$ is lower semi-continuous and *linear* in P; (ii) $\{P : H(P|Q_0) \leq a\}$ is a compact set for any a. See section 3 in [DV2] for details.

Lemma 2.1 *Consider a continuous function F on $\mathbf{\Omega}$ measurable w.r.t. paths up to time T. Let also F satisfy $E^{Q_0}e^F \leq 1$. Then for any $t > 0$,*

$$E^{Q_0} \exp\{\frac{1}{T}\int_0^t F(\omega(s+\cdot) - \omega(s))\, ds\} \leq 1.$$

A proof follows the proof of lemma 4.1 in [DV2] closely. We skip details.

For any $\omega(\cdot)$, denote by $J(t,\omega)$ the number of jumps in the path over the time interval $[0,t]$. Note importantly that $\{P \in \mathbf{P_{si}(\Omega)} : E^P J(1,\omega) \leq a\}$ is a compact set for any a

Lemma 2.2 *We have the following super exponential tightness:*

$$\limsup_{t\to\infty} \frac{1}{t} \ln E^{Q_0} \exp\{tE^{L_t(\cdot,\omega)} J(1,\omega')\} < \infty. \tag{2.4}$$

Proof: Note that $J(1,\omega)$ is a Poisson variable with unit intensity. A simple calculation yields

$$E^{Q_0}\exp\{J(1,\omega)\} = e^{e-1}.$$

The claim follows by combining the above equation and Lemma 2.1 and □

The following limit is the one of the key steps to deriving our main results. Due to Varadhan's theorem (see [V1]), we may state the following corollary to our principle; note in the specified domains of integration that $w_t(\cdot) = w(\cdot)$:

Corollary 2.1 *Assume f is as defined in Theorem 1.1. For any $T > 0$, we have*

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\ln E^{Q_0}\exp\{-tf[\frac{1}{t}\int_T^{t-T}|\lambda|^\alpha(\int_{-T}^{T}\delta_0(\omega(s+r)-\omega(s))\,dr)^{\alpha-1}\,ds\}\\ = -\inf_{\{P\in\mathbf{P_{si}(\Omega)}\}}\{f(|\lambda|^\alpha E^P(\int_{-T}^{T}\delta_0(\omega(r))\,dr)^{\alpha-1})+H(P|Q_0)\}.\end{aligned}\tag{2.5}$$

3 Decay Rates: Proof of Theorem 1.1

Our intention in this section is to prove Theorem 1.1 by applying the large deviation principle outlined earlier. The first step is to rewrite the Hamiltonian $H(t;\lambda;f)$. It isn't hard to see that

$$\sum_{x\in\mathbf{Z^d}}(\int_0^t\delta_x(\omega(s))\,ds)^\alpha = \int_0^t(\int_0^t\delta_0(\omega(r)-\omega(s))\,dr)^{\alpha-1}\,ds.$$

We will prove Theorem 1.1 in several steps. First we consider the upper bound:

Lemma 3.1 *Assume that f, α, λ and $E(t;\lambda;f)$ are as defined in Theorem 1.1. Then*

$$\limsup_{t\to\infty}E(t;\lambda;f)\le-\inf_{P\in\mathbf{P_{si}(\Omega)}}\{f(\lambda^\alpha E^P(\int_{-\infty}^{\infty}\delta_0(\omega(s))\,ds)^{\alpha-1})+H(P|Q_0)\}.\tag{3.1}$$

Proof: Observe that for T such that $0<2T<t$ we have

$$\begin{aligned}f(\frac{1}{t}\sum_{x\in\mathbf{Z^d}}(\lambda\int_0^t\delta_x(\omega_s)\,ds)^\alpha) &= f(\frac{1}{t}\int_0^t\lambda^\alpha(\int_0^t\delta_0(\omega(r)-\omega(s))\,dr)^{\alpha-1}\,ds)\\ &\ge f(\frac{1}{t}\int_T^{t-T}\lambda^\alpha(\int_{-T}^{T}\delta_0(\omega(r+s)-\omega(s))\,dr)^{\alpha-1}\,ds).\end{aligned}$$

Therefore

$$E(t;\lambda;f)\le\frac{1}{t}\ln E^{Q_0}[\exp\{-tf(\frac{1}{t}\int_T^{t-T}\lambda^\alpha(\int_{-T}^{T}\delta_0(\omega(s+r)-\omega(r))\,ds)^{\alpha-1}\,dr)\}].$$

Now send $t \to \infty$. As an immediate consequence of Corollary2.1 we obtain

$$\limsup_{t\to\infty} E(t;\lambda;f) \le -\inf_{P\in\mathbf{P_{si}(\Omega)}} \{f(\lambda^\alpha E^P(\int_{-T}^{T} \delta_0(\omega(s))\, ds)^{\alpha-1}) + H(P|Q_0)\}.$$

What remains is to check

$$\begin{aligned}&\lim_{T\to\infty} \inf_P \{f(\lambda^\alpha E^P(\int_{-T}^{T} \delta_0(\omega(s))\, ds)^{\alpha-1}) + H(P|Q_0)\}\\ &= \inf_P \{f(\lambda^\alpha E^P(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds)^{\alpha-1}) + H(P|Q_0)\}.\end{aligned}$$

This is done in the next lemma. This completes the proof. □

We now derive several properties for our variational problem. Let us call the T-truncated version of the full variational quantity‘ $S_c(\lambda,\alpha,f)$ as

$$S^T(\lambda,\alpha,f) = \inf_{P\in\mathbf{P_{si}(\Omega)}} \{f(\lambda^\alpha E^P(\int_{-T}^{T} \delta_0(\omega(s))\, ds)^{\alpha-1}) + H(P|Q_0)\}.$$

Lemma 3.2 *The following properties are valid:*

1. *There exists an ergodic minimizer P_0 such that $H(P_0|Q_0) < \infty$.*
2. $\lim_{T\to\infty} S^T(\lambda,\alpha,f) = S_c(\lambda,\alpha,f)$.

Proof:

Step 1: Consider the object

$$G(P) = f(\lambda^\alpha E^P(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds)^{\alpha-1}) + H(P|Q_0).$$

We note that $G(P)$ is a nonnegative concave function in P due to the linearity of $H(P|Q_0)$ in P and the concavity of f.

Now for each $0 < \mu < \frac{1}{d}$, define Q_μ as the law of $W_\mu(t)$, a nonsymmetric random walk on $\mathbf{Z^d}$:

$$W_\mu(t) = \sum_{i=1}^{A(t)} Y_i.$$

Here $A(t)$ is a Poisson process with intensity $2d$ and $\{Y_i : i = 1, 2, \cdots\}$ is a sequence of i.i.d random vectors with common distribution $Pr\{Y_1 = e_k\} = 1/2d$ for $2 \le k \le d$, $Pr\{Y_1 = e_1\} = 1/2d + \mu/2$ and $Pr\{Y_1 = -e_1\} = 1/2d - \mu/2$; $\{e_k : k = 1, \cdots, d\}$ is the standard basis on $\mathbf{Z^d}$.

It is not hard to check that $G(Q_\mu) < \infty$. This implies that $S_c(\lambda,\alpha,f) < \infty$.

Let P_n be a minimizing sequence selected to satisfy $G(P_n) < G(Q_\mu) + 1$. Hence $H(P_n|Q_0) < G(Q_\mu)+1$. Immediately we have that $\{P_n : n = 1, 2.\cdots\}$ is tight. Assume

P_0 to be a limit point of the sequence. Without trouble, we may conclude that P_0 is a minimizer. As $G(P)$ is concave in P, the set of minimizers is a convex set, and therefore we may take P_0 to be an extreme point, hence ergodic.

Step 2:

To simply expression, we will drop λ, α, and f dependencies from our notation. Obviously $S^T \le S_c(\lambda)$, for any $T > 0$. Let $T \to \infty$; we then have $\lim_{T\to\infty} S^T \le S_c(\lambda)$.

What remains is to check the lower bound. To this end, define the truncated function

$$G_T(P) = f(\lambda^\alpha E^P(\int_{-T}^{T} \delta_0(\omega(s))\, ds)^{\alpha-1}) + H(P|Q_0).$$

Now choose a $P_T \in \mathbf{P_{si}}(\otimes)$ such that

$$S^T \ge G_T(P_T) - \frac{1}{1+T}.$$

This implies that $\sup_T H(P_T|Q_0) < \infty$ and therefore the tightness of $\{P_T\}$. Let P_1 be a limit point of the set. Without loss of generality, we may assume $P_T \to P_1$. Then for any $\tau > 0$, we have

$$\begin{aligned} \liminf_{T\to\infty} S^T &\ge \liminf_{T\to\infty} G_\tau(P_T) \\ &= G_\tau(P_1). \end{aligned}$$

At this point, allow $\tau \to \infty$, to obtain $\lim_{T\to\infty} S^T \ge G(P_1) \ge S_c$ completing the proof. □

We now derive the lower bound in Theorem 1.1.

Lemma 3.3 *Let f, α, λ and $E(t;\lambda;f)$ be as before in Theorem 1.1, then*

$$\liminf_{t\to\infty} E(t) \ge -\inf_{P\in\mathbf{P_{si}}(\mathbf{\Omega})} \{f(|\lambda|^\alpha E^P(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds)^{\alpha-1}) + H(P|Q_0)\}.$$

Proof: Let $P \in \mathbf{P_{si}}(\mathbf{\Omega})$ be such that $H(P|Q_0) < \infty$. For such P, consider the Radon-Nikodym derivative $dP/dQ_0(t)$ with respect to σ-fields generated by $\{w(s) : 0 \le s \le t\}$. We write

$$\begin{aligned} E(t;\lambda;f) &= \frac{1}{t}\ln E^{Q_0}\exp\{-tf(\frac{1}{t}\int_0^t \lambda^\alpha(\int_0^t \delta_0(\omega(r)-\omega(s))\, dr)^{\alpha-1}\, ds)\} \\ &= \frac{1}{t}\ln E^{P}\exp\{-tf(\frac{1}{t}\int_0^t \lambda^\alpha(\int_0^t \delta_0(\omega(r)-\omega(s))\, dr)^{\alpha-1}\, ds) \\ &\qquad -\ln\frac{dP}{dQ_0}(t)\} \end{aligned}$$

$$
\begin{aligned}
&\geq -E^P f(\frac{1}{t}\int_0^t \lambda^\alpha(\int_0^t \delta_0(\omega(r)-\omega(s))\,dr)^{\alpha-1}\,ds) - \frac{1}{t}E^P \ln\frac{dP}{dQ_0}(t)\\
&\geq -f(\frac{1}{t}\int_0^t \lambda^\alpha E^P(\int_0^t \delta_0(\omega(r)-\omega(s))\,dr)^{\alpha-1}\,ds) - \frac{1}{t}E^P \ln\frac{dP}{dQ_0}(t)\\
&\geq -f(\frac{1}{t}\int_0^t \lambda^\alpha E^P(\int_{-\infty}^\infty \delta_0(\omega(r)-\omega(s))\,dr)^{\alpha-1}\,ds) - \frac{1}{t}E^P \ln\frac{dP}{dQ_0}(t)\\
&= -f(\lambda^\alpha E^P(\int_{-\infty}^\infty \delta_0(\omega(r))\,dr)^{\alpha-1}) - \frac{1}{t}E^P \ln\frac{dP}{dQ_0}(t).
\end{aligned}
$$

In the above series, we've applied the property that f is concave in the fourth step, that f increases in the fifth step and that P has stationary increments in the last step.

To finish the argument let $t \to \infty$. On account of that

$$
E^P[\ln dP/dQ_0(t)] \to H(P|Q_0),
$$

we have

$$
\liminf_{t\to\infty} E(t;\lambda;f) \geq -f(\lambda^\alpha E^P(\int_{-\infty}^\infty \delta_0(\omega_r)\,dr)^{\alpha-1}) - H(P|Q_0).
$$

This completes the proof. □

4 Asymptotic Behavior of S_c in $d \geq 3$

In this section, we will address the part of Theorem 1.2 concerning transient dimensions $d \geq 3$.

Step 1: We establish the upper bound first. Note trivially that $H(Q_0|Q_0) = 0$. Also on account of transience, the number of visits to 0 is Q_0 almost surely finite: $E^{Q_0}\int_{-\infty}^\infty \delta_0(\omega(s))\,ds < \infty$. Therefore by Holder's inequality, as $0 < \alpha - 1 \leq 1$,

$$
E^{Q_0}[\int_{-\infty}^\infty \delta_0(\omega(s))\,ds]^{\alpha-1} < \infty.
$$

It is straightforward now test Q_0 itself into the infimum to obtain

$$
\begin{aligned}
S_c(\lambda,\alpha,x^\beta) &\leq (\lambda^\alpha E^{Q_0}(\int_{-\infty}^\infty \delta_0(\omega(s))\,ds)^{\alpha-1})^\beta + H(Q_0|Q_0)\\
&\leq \lambda^{\alpha\beta}(E^{Q_0}(\int_{-\infty}^\infty \delta_0(\omega(s))\,ds)^{\alpha-1})^\beta.
\end{aligned}
$$

Hence $\limsup_{\lambda\to 0} S_c(\lambda,\alpha,\beta)/\lambda^{\alpha\beta} \leq (E^{Q_0}(\int_{-\infty}^\infty \delta_0(\omega(s))\,ds)^{\alpha-1})^\beta$.

Step 2: The lower bound follows now. For the variational quantity

$$
\inf_P\{[\lambda^\alpha E^P(\int_{-\infty}^\infty \delta_0(\omega(s))\,ds)^{\alpha-1}]^\beta + H(P|Q_0)\},
$$

let P_λ be a minimizer. Recall $G(P)$ from the proof of Lemma3.2. Then $G(P_\lambda) = S_c(\lambda,\alpha,\beta)$.

It follows from Step 1 that $\lim_{\lambda\to 0} G(P_\lambda) = 0$. Hence $\lim_{\lambda\to 0} H(P_\lambda|Q_0) = 0$ implying that P_λ converges weakly to Q_0. Observe now that for any $T > 0$

$$\begin{aligned} S_c(\lambda,\alpha,\beta) &= G(P_\lambda) \\ &\geq \lambda^{\alpha\beta}(E^{P_\lambda}(\int_{-T}^{T} \delta_0(\omega(s))\, ds)^{\alpha-1})^\beta. \end{aligned}$$

Consequently

$$\begin{aligned} \liminf_{\lambda\to 0} \frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\alpha\beta}} &\geq \lim_{\lambda\to 0} (E^{P_\lambda}(\int_{-T}^{T} \delta_0(\omega(s))\, ds)^{\alpha-1})^\beta \\ &= (E^{Q_0}(\int_{-T}^{T} \delta_0(\omega(s))\, ds)^{\alpha-1})^\beta. \end{aligned}$$

At this stage, we send $T \to \infty$, obtaining

$$\liminf_{\lambda\to 0} \frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\alpha\beta}} \geq (E^{Q_0}(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds)^{\alpha-1})^\beta$$

to complete the proof. □

5 Asymptotic Behavior of S_c in $d = 1, 2$

In this section, we complete the proof of Theorem 1.2 by giving arguments for the second half of theorem concerning the recurrent dimensions $d = 1, 2$. Our proof will consist of two parts. The first is to check the upper bound. And the second is to derive the lower bound. The upper estimate is simpler than its lower companion as we need only select a suitable test measure P to substitute into our variational formula derived earlier for S_c in Theorem 1.1. For the lower bound, more carefully the entropy inequality is used.

Upper Bounds: We begin with the following lemma:

Lemma 5.1 *Let Q_μ be as defined in the proof of Theorem 3.2. Then*

$$\lim_{\mu\downarrow 0} \frac{E^{Q_\mu}\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds}{1/\mu} = 1 \tag{5.1}$$

when $d = 1$.

$$\lim_{\mu\downarrow 0} \frac{E^{Q_\mu}\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds}{\ln(1/\mu)} = \frac{2}{\pi} \tag{5.2}$$

when $d = 2$.

A proof is a straightforward calculation by using Fourier integral representation for transition probabilities of rondom walk in [Sp]. We skip details

Lemma 5.2 *We have in $d = 2$ that*

$$\limsup_{\lambda\to 0} \frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\alpha\beta}(\ln\frac{1}{\lambda})^{(\alpha-1)\beta}} < \infty.$$

And in $d = 1$ we have

$$\limsup_{\lambda\to 0} \frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\frac{2\alpha\beta}{2+\beta(\alpha-1)}}} < \infty.$$

Proof: Recall the object $G(P)$ from section 3:

$$G(P) = (\lambda^\alpha E^P(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds)^{\alpha-1})^\beta + H(P|Q_0).$$

Clearly we have that $S_c(\lambda,\alpha,\beta) \le G(P)$ for all $P \in \mathbf{P_{si}}(\mathbf{\Omega})$.

Consider for $d = 2$ the measure $P = Q_{\lambda^{\frac{\alpha\beta}{2}}}$; and for $d = 1$ relatedly $P = Q_{\lambda^{\frac{\alpha\beta}{2+\beta(\alpha-1)}}}$. In view of Lemmas 5.1 and well known fact

$$\lim_{\mu\downarrow 0} \frac{H(Q_\mu|Q_0)}{\mu^2} = \frac{d}{2},$$

and Holder's inequality using $0 < \alpha - 1 \le 1$, we obtain the upper estimates (5.2) and (5.2) respectively. This completes the proof. □

Remark: In the case $d = 2$, the above argument gives exactly

$$\limsup_{\lambda\to 0} \frac{S_c(\lambda,2,1)}{\lambda^2 \ln\frac{1}{\lambda}} = \frac{2}{\pi}.$$

Lower Bounds:

The following lemma is the crucial step in our argument.

Lemma 5.3 *For any $t > 0$,*

$$S_c(\lambda,\alpha,\beta) \ge -\frac{1}{2t} \ln E^{Q_0} \exp\{-2t\lambda^{\alpha\beta}[\int_{-t}^{t} \delta_0(\omega(s))\, ds]^{(\alpha-1)\beta}\}. \tag{5.3}$$

Proof: Observe that for each $t > 0$ and $P \in \mathbf{P_{si}}(\mathbf{\Omega})$

$$\begin{aligned}[|\lambda|^\alpha E^P(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds)^{\alpha-1}]^\beta &\ge \lambda^{\beta\alpha} E^P(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds)^{(\alpha-1)\beta} \\ &\ge \lambda^{\beta\alpha} E^P(\int_{-t}^{t} \delta_0(\omega(s))\, ds)^{(\alpha-1)\beta}.\end{aligned}$$

At this point, we apply the entropy inequality to the object

$$-2t\lambda^{\alpha\beta}(\int_{-t}^{t} \delta_0(\omega(s))\, ds)^{(\alpha-1)\beta}.$$

Consequently we obtain

$$\begin{aligned} & -E^P 2t\lambda^{\alpha\beta}(\int_{-t}^{t} \delta_0(\omega(s))\, ds)^{(\alpha-1)\beta} \\ \leq \quad & \ln E^{Q_0} \exp\{-2t\lambda^{\alpha\beta}[\int_{-t}^{t} \delta_0(\omega(s))\, ds]^{(\alpha-1)\beta}\} + H_t(P|Q_0). \end{aligned}$$

Hence

$$\begin{aligned} & [\lambda^{\alpha} E^P(\int_{-\infty}^{\infty} \delta_0(\omega(s))\, ds)^{\alpha-1}]^{\beta} + \frac{1}{2t} H_t(P|Q_0) \\ \geq \quad & -\frac{1}{2t} \ln E^{Q_0} \exp\{-2t\lambda^{\alpha\beta}[\int_{-t}^{t} \delta_0(\omega(s))\, ds]^{(\alpha-1)\beta}\}. \end{aligned}$$

Now use the fact that

$$\frac{1}{2t} H_t(P|Q_0) \uparrow H(P|Q_0).$$

to complete the proof. □

The next lemma, a beautiful classical limit theorem(so called Kallianpur-Robbins law) regarding occupation times of random walk, is helpful to us. We only state it here; for a proof of the result and an interesting summary of its early history see [DK] (especially Theorem 1).

Lemma 5.4 *For $d = 1$*

$$\lim_{r\to\infty} Q_0\{\omega : \frac{2}{\sqrt{r}} \int_0^r \delta_0(\omega(s))\, ds < a\} = (\pi)^{-\frac{1}{2}} \int_0^a e^{-\frac{y^2}{4}}\, dy. \tag{5.4}$$

For $d = 2$

$$\lim_{r\to\infty} Q_0\{\omega : \frac{2\pi}{\ln r} \int_0^r \delta_0(\omega(s))\, ds < a\} = 1 - e^{-a}. \tag{5.5}$$

In the following lemma, we finish the proof of the lower estimates, and therefore we remark, the proof of Theorem 1.2.

Lemma 5.5 *For $d = 2$ we have*

$$\liminf_{\lambda\to 0} \frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\alpha\beta}(\ln\frac{1}{\lambda})^{(\alpha-1)\beta}} > 0.$$

For $d = 1$ we have

$$\liminf_{\lambda\to 0} \frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\frac{2\alpha\beta}{2+\beta(\alpha-1)}}} > 0.$$

Proof: From Lemma 5.3, we have

$$S_c(\lambda,\alpha,\beta) \geq -\frac{1}{2t} \ln E^{Q_0} \exp\{-2t\lambda^{\alpha\beta}[\int_{-t}^{t} \delta_0(\omega(s))\, ds]^{(\alpha-1)\beta}\}. \tag{5.6}$$

In the case $d = 2$ select

$$t = \frac{1}{\lambda^{\alpha\beta}(\ln\frac{1}{\lambda})^{(\alpha-1)\beta}}.$$

Substituting now into (5.6) we have

$$\frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\alpha\beta}(\ln\frac{1}{\lambda})^{(\alpha-1)\beta}} \geq -\frac{1}{2} \ln E^{Q_0} \exp\{-[\frac{2}{\ln\frac{1}{\lambda}} \int_{-\frac{1}{\lambda^{\alpha\beta}(\ln\frac{1}{\lambda})^{(\alpha-1)\beta}}}^{\frac{1}{\lambda^{\alpha\beta}(\ln\frac{1}{\lambda})^{(\alpha-1)\beta}}} \delta_0(\omega(s))\, ds]^{(\alpha-1)\beta}\}.$$

Understanding the convergence in (5.5) gives the result

$$\liminf_{\lambda\to 0} \frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\alpha\beta}(\ln\frac{1}{\lambda})^{(\alpha-1)\beta}} > 0.$$

For the case $d = 1$ choose

$$t = \frac{1}{\lambda^{\frac{2\alpha\beta}{2+\beta(\alpha-1)}}}.$$

The same argument as for the case $d = 2$, but now using (5.4), yields

$$\liminf_{\lambda\to 0} \frac{S_c(\lambda,\alpha,\beta)}{\lambda^{\frac{2\alpha\beta}{2+\beta(\alpha-1)}}} > 0.$$

This completes our proof. □.

Remark: When $d = 2$, the above argument gives exactly

$$\liminf_{\lambda\to 0} \frac{S_c(\lambda,2,1)}{\lambda^2 \ln\frac{1}{\lambda}} = \frac{2}{\pi}.$$

We state as a consequence of the two remarks after Lemmas 5.2 and 5.5 respectively, the limit:

Corollary 5.1 *In the case $d = 2$*

$$\lim_{\lambda\to 0} \frac{S_c(\lambda,2,1)}{\lambda^2 \ln\frac{1}{\lambda}} = \frac{2}{\pi}.$$

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