LARGE DEVIATIONS FOR A TAGGED PARTICLE IN 1D NEAREST-NEIGHBOR SYMMETRIC SIMPLE EXCLUSION

S. SETHURAMAN AND S.R.S. VARADHAN

Abstract. Laws of large numbers, starting from certain nonequilibrium measures, have been shown recently for a tagged particle in one dimensional symmetric nearest-neighbor simple exclusion (Jara-Landim (2006)). In this article, we prove a corresponding large deviation principle, and evaluate the rate function in certain regimes, showing some phase transitions.

1. Introduction and Results

The one dimensional nearest-neighbor symmetric simple exclusion process follows a collection of random walks on the lattice $\mathbb{Z}$ which move equally likely to the nearest left or right independently except in that jumps to already occupied sites are suppressed. More precisely, the model is a Markov process $\eta_t = \{\eta_t(x) : x \in \mathbb{Z}\}$ evolving on the configuration space $\Sigma = \{0, 1\}^{\mathbb{Z}}$ with generator,

$$(L\phi)(\eta) \quad = \quad \frac{1}{2}\sum_x \big[\eta(x)(1-\eta(x+1)) + \eta(x+1)(1-\eta(x))\big]\big(\phi(\eta^{x,x+1}) - \phi(\eta)\big)$$

where $\eta^{x,y}$, for $x \neq y$, is the configuration obtained from η by exchanging the values at x and y,

$$\eta^{x,y}(z) \;=\; \begin{cases} \eta(z) & \text{when } z \neq x, y \\ \eta(x) & \text{when } z = y \\ \eta(y) & \text{when } z = x. \end{cases}$$

A detailed treatment can be found in Liggett [20].

As the process is 'mass conservative,' that is no birth or death, one expects a family of invariant measures corresponding to particle density. In fact, for each $\rho \in [0, 1]$, the product over $\mathbb{Z}$ of Bernoulli measures ν_ρ which independently puts a particle at locations $x \in \mathbb{Z}$ with probability ρ, that is $\nu_\rho(\eta_x = 1) = 1 - \nu_\rho(\eta_x = 0) = \rho$, are invariant. We will denote E_ρ as expectation under ν_ρ.

Consider now a distinguished, or tagged particle, say initially at the origin. Let X_t be its position at time t. The problem of characterizing the asymptotic behavior of X_t in interacting systems has a long history (cf. Spohn [32, Chapters 8.I, 6.II]), and was mentioned in Spitzer's seminal paper [31].

The goal of this paper is to study the large deviations of X_t when the initial distribution of particles is part of a large class of nonequilibrium measures. Part of our motivation is that recently laws of large numbers (LLN) and central limit

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theorems (CLT) starting from a class of 'local equilibrium' initial measures have been proved in Jara-Landim [13].

The article [13] is a significant nonequilibrium generalization of Arratia's CLT [1] which established 'subdiffusive' behavior in the 1D nearest-neighbor symmetric simple exclusion model. Namely, starting under an equilibrium $\nu_\rho(\cdot|\eta(0)=1)$, $t^{-1/4}X_t \Rightarrow \mathrm{N}(0,\sigma^2)$, where $\sigma^2 = \sqrt{2/\pi}(1-\rho)/\rho$. Physically, the 'subdiffusive' scale in the CLT is explained as due to 'trapping' induced from the nearest-neighbor dynamics which enforces a rigid ordering of particles. Other proofs of this CLT are found in Rost-Vares [26] and De Masi-Ferrari [8]. Recently, the CLT was extended to an invariance principle with respect to a fractional Brownian motion, $\lambda^{1/4}X_{\lambda t} \Rightarrow \sigma fBM_{1/4}(t)$, in Peligrad-Sethuraman [23].

We now specify the class of initial measures considered, that is 'deterministic initial configurations' and 'local equilibrium product measures.' Let M_1 be the space of functions $\gamma:\mathbb{R}\rightarrow[0,1]$, and let $M_1(\rho_*,\rho^*)$ be those functions in M_1 which equal ρ_* for all $x\leq x_*$, and which equal ρ^* for all $x\geq x^*$, for some $x_*\leq x^*$.

We will consider on M_1 the topology induced by $C_K(\mathbb{R})$, the set of continuous compactly supported functions on $\mathbb{R}$, with the duality $\langle\cdot;\cdot\rangle$ where $\langle\gamma;G\rangle = \int G(x)\gamma(x)dx$ for $\gamma\in M_1$ and $G\in C_K(\mathbb{R})$. This topology, if M_1 is thought of as a measure space, is the vague topology which is metrizable.

Local equilibrium measure (LEM). For $0<\rho_*,\rho^*<1$, let $\gamma\in M_1(\rho_*,\rho^*)$ be an a.s. (Lebesgue) continuous function such that $0<\gamma(x)<1$ for all $x\in\mathbb{R}$. With respect to γ and a scaling parameter $N\geq 1$, we define a sequence of local equilibrium product measures $\nu^{(N)}_{\gamma(\cdot)}$ as those formed from the marginals $\nu^{(N)}_{\gamma(\cdot)}(\eta(x)=1) = \gamma(x/N)$ for $x\neq 0$, and $\nu^{(N)}_{\gamma(\cdot)}(\eta(0)=1)=1$.

Deterministic initial configuration (DIC). For $0<\rho_*,\rho^*<1$, let γ be an a.s. (Lebesgue) continuous function in $M_1(\rho_*,\rho^*)$. Then, a sequence of deterministic initial configurations $\xi^{\gamma,N}$ is one such that $\xi^{\gamma,N}(0)=1$ and, for all continuous, compactly supported G, $\lim_{N\rightarrow\infty}\frac{1}{N}\sum_x \xi^{\gamma,N}(x)G\big(x/N\big) = \int G(x)\gamma(x)dx$.

We remark particular examples of local equilibrium measures $\nu^{(N)}_{\gamma(\cdot)}$ are the equilibrium measures $\nu_\rho(\cdot|\eta(0)=1)$ conditioned to have a particle at the origin for $0<\rho<1$. Suitable deterministic configurations $\xi^{\gamma,N}$ for instance include the 'alternating' configuration where every other vertex is occupied corresponding to $\gamma(x)\equiv 1/2$. Nonequilibrium initial measures corresponding to step profiles $\gamma(x) = \rho_* 1_{(-\infty,0]}(x) + \rho^* 1_{(0,\infty)}(x)$, for $\rho_*,\rho^*\in(0,1)$, and other piecewise continuous profiles can also be constructed.

In a sense, the profiles γ associated to the local equilibria and deterministic profiles above are 'non-degenerate' in that γ is asymptotically bounded strictly between 0 and 1. Also, the property that $\gamma(x)$ is constant for large $|x|$, and with respect to (LEM) specifications that $0<\gamma<1$, is useful in the proofs of later Propositions 1.3 and 1.5, although some modifications, for instance in terms of profiles sufficiently close to being constant for large $|x|$, should be possible with more work. Other comments about some 'degenerate' profiles, under which different tagged particle large deviation behaviors arise, are made after Theorem 1.6.

We now describe the LLN proved in Jara-Landim [13] (stated under a class of local equilibrium measures, but the same proof also works starting from the initial measures above):

$$\lim_{N\to\infty} \frac{1}{N} X_{N^2 t} \;=\; u_t, \tag{1.1}$$

in probability, the solution of

$$\frac{du_t}{dt} = -\frac{1}{2}\frac{\partial_u \rho(t,u_t)}{\rho(t,u_t)}$$

where $\partial_t \rho = (1/2)\partial_{xx}\rho$ and $\rho(0,x) = \gamma(x)$, that is $\rho(t,x) = \sigma_t * \gamma(x)$ where $\sigma_t(y) = (2\pi t)^{-1/2}\exp\{-y^2/2t\}$ Note that u_t is also the unique number α where

$$\int_0^\alpha \rho(t,u)du = -\frac{1}{2}\int_0^t (\partial_u \rho)(s,0)ds = \int_0^\infty \big[\rho(t,x) - \rho(0,x)\big]dx. \tag{1.2}$$

In this equation, the right-side is the integrated macroscopic current across the origin up to time t. As the microscopic dynamics is nearest-neighbor with enforced ordering of particles, the tagged particle, initially at the origin, will be at the head of the flow through the origin. So, to compute its macroscopic position u_t at time t, we find α so that the mass at time t between positions $x = 0$ and $x = \alpha$, the left-side of the equation, equals the integrated current, and conclude $u_t = \alpha$.

We remark, starting from local equilibrium measures, a corresponding invariance principle in subdiffusive $t^{1/4}$ scale as in Arratia's CLT, in the sense of finite-dimensional distributions, with respect to a fractional Brownian motion-type Gaussian process was also proved in [13]. Also, starting from an equilibrium ν_ρ, the result $X_N/N \to 0$, is a special case of a LLN limit, for more general exclusion processes, in Saada [27].

In this context, we derive a large deviation principle (LDP) (Theorem 1.6), in diffusive scale, corresponding to the law of large numbers (1.1) when starting from (LEM) or (DIC) measures. We show also that the associated rate function, starting from deterministic initial configurations with $\gamma(x) \equiv \rho$ has a certain phase transition: Namely, it is quadratic near its zero, but is third order far away from the zero (Theorems 1.7, 1.8). On the other hand, starting from a 'degenerate' deterministic initial configuration with $\gamma(x) = 1_{[-1,1]}(x)$, we show the large deviations behavior is at most quadratic (Theorem 1.9).

The main idea for the LDP is to relate, through several 'entropy' and 'energy' estimates, the tagged particle deviations to those established in Kipnis-Olla-Varadhan [16], Landim [17], and Landim-Yau [19] with respect to the hydrodynamic limit of the process empirical density (cf. Propositions 1.1, 1.4). The phase transition asymptotics are proved in part by estimations of currents and calculus of variations arguments.

At this point, we remark that the behavior of a tagged particle, in contrast to Arratia's result, scales differently in symmetric exclusion models in $d \geq 2$, and also in $d = 1$ when the underlying jump probability is not nearest-neighbor, that is when particles are free to pass by other particles. Namely, in Kipnis-Varadhan [15], starting under an equilibrium $\nu_\rho(\cdot|\eta(0) = 1)$, in diffusive scale, an invariance principle for the tagged particle to a Brownian motion was proved. Later, in Rezakhanlou [25], starting from local equilibrium measures, in diffusive scale, an invariance principle with respect to a diffusion with a drift given in terms of the profile γ is proved for the 'averaged' tagged particle position, averaging over all the positions of $O(N)$ particles in a sequence of tori with N vertices. In Quastel-Rezakhanlou-Varadhan [22], in $d \geq 3$, a corresponding large deviations principle is proved for the 'averaged' tagged particle position with rate function which is finite on processes with finite relative entropy with respect to diffusions which in some sense add an additional drift to the limit diffusion in [25]. This LDP would seem also to hold in $d \leq 2$ given regularity results on the self-diffusion coefficient in Landim-Olla-Varadhan [18] not available when [22] was written.

We also observe the third order asymptotics we derive for the tagged particle rate function in Theorem 1.7 parallel formal third order expansions for the probability distribution of the current across the origin at large times in Derrida-Gerschenfeld [10]. This is discussed a bit more after Theorem 1.7.

We also mention, other large deviation works with respect to empirical densities and currents in related interacting systems are Benois-Landim-Kipnis [3], Bertini-DeSole-Gabrielli-Jona Lasinio-Landim [4], [5], Bertini-Landim-Mourragui [6], Farfan-Landim-Mourragui [11], and Grigorescu [12]; see also Kipnis-Landim [14][Chapter 10] and references therein. Also, we note, with respect to totally asymmetric nearest-neighbor exclusion in $d = 1$, large deviation 'lower tail' bounds for tagged particles are found in Seppäläinen [29].

We now give the hydrodynamic limit and rate function for the process empirical density $\mu^N(s, x; \eta) \in D([0, T]; M_1)$,

$$\mu^N(s, x; \eta) = \sum_{k\in\mathbb{Z}} \eta_{N^2 s}(k) 1_{[k/N,(k+1)/N)}(x)$$

where $x \in \mathbb{R}, s \in [0, T]$, and $0 < T < \infty$ is a fixed time.

Proposition 1.1. *Starting from local equilibrium measures, or deterministic configurations, we have, for $t \in [0, T]$, $\epsilon > 0$ and smooth, compactly supported ϕ, that*

$$\lim_{N\uparrow\infty} P\Big\{\Big| \int \phi(x)\mu^N(t, x)dx - \int \phi(x)m(t, x)dx\Big| > \epsilon\Big\} \;=\; 0$$

where m satisfies $\partial_t m = (1/2)\partial_{xx} m$ with initial data $m(0, x) = \gamma(x)$.

A reference for the proof of Proposition 1.1, among other places, is Theorem 8.1 Seppäläinen [30].

The rate functions for the process empirical density differ depending on the type of initial distribution. First, following [16], [17], suppose the process starts from a local equilibrium measure $\nu^{(N)}_{\gamma(\cdot)}$. For $\mu \in D([0, T]; M_1)$, define the linear functional on $C^{1,2}_K([0, T] \times \mathbb{R})$:

$$\begin{aligned} l(\mu; G) \;&=\; \int G(T, x)\mu_T(x)dx - \int G(0, x)\mu_0(x)dx \\ &\quad - \int_0^T \int \mu_t(x)\Big(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\Big)G(t, x)dxdt. \end{aligned}$$

Let

$$\begin{aligned} I_0(\mu) \;&=\; \sup_{G\in C^{1,2}_K([0,T]\times\mathbb{R})} \Big\{ l(\mu; G) - \frac{1}{2}\int_0^T \int \mu_t(1 - \mu_t)(x)G_x^2(t, x)dxdt\Big\}, \\ h(\mu_0; \gamma) \;&=\; \sup_{\phi_0,\phi_1\in C_K(\mathbb{R})} \Big\{ \int \mu_0(x)\phi_0(x)dx + \int (1 - \mu_0)(x)\phi_1(x)dx \\ &\qquad - \int \log[\gamma(x)e^{\phi_0(x)} + (1 - \gamma(x))e^{\phi_1(x)}]dx\Big\}, \end{aligned}$$

and form the rate function

$$I_\gamma^{\rm LE}(\mu) = I_0(\mu) + h(\mu_0;\gamma).$$

Here, $C_K^{\alpha,\beta}$ is the space of compactly supported functions, α and β-times continuously differentiable in t and x respectively. In addition, we will use the notation $\mu_t(x) = \mu(t,x)$.

Next, starting from deterministic configurations $\xi^{\gamma,N}$, the rate function in [19] (given for zero-range systems, but the methods straightforwardly apply to our exclusion context) is given by

$$I_\gamma^{\rm DC}(\mu) \ = \ \left\{ \begin{array}{rl} I_0(\mu) & \text{when } \mu_0 = \gamma \\ \infty & \text{otherwise.} \end{array} \right\}$$

To simplify notation, we call both I_γ^{LE} and $I_\gamma^{\rm DC}$ as I_γ, omitting the super scripts 'LE' and 'DC,' when statements apply to both and the context clear. For $0 \le \alpha, \beta \le 1$, let $h_d(\alpha;\beta) = \alpha\log[\alpha/\beta] + (1-\alpha)\log[(1-\alpha)/(1-\beta)]$ with usual conventions $0\log 0 = 0/0 = 0$ and $\log 0 = -\infty$.

A main point in [16] was to note that when $I_\gamma(\mu) < \infty$ is finite that first

$$h(\mu_0;\gamma) \ = \ \int h_d(\mu_0(x);\gamma(x))dx \ < \ \infty.$$

[Of course, starting from deterministic configurations, $\mu_0 = \gamma$.] Also second, μ corresponds to a function $H_x \in L^2([0,T]\times\mathbb{R}, \mu(1-\mu)dxdt)$ and satisfies a 'weakly asymmetric hydrodynamic equation,'

$$\partial_t\mu \ = \ \frac{1}{2}\partial_{xx}\mu - \partial_x\big[H_x\mu(1-\mu)\big] \tag{1.3}$$

in the weak sense. That is, for $G \in C_K^{1,2}([0,T]\times\mathbb{R})$, we have

$$l(\mu;G) \ = \ \int_0^T\int G_xH_x\mu(1-\mu)(t,x)dtdx, \tag{1.4}$$

and

$$I_0(\mu) \ = \ \frac{1}{2}\int_0^T\int H_x^2\mu(1-\mu)dxdt. \tag{1.5}$$

Reciprocally, if for a density $\mu \in D([0,T];M_1)$ there exists $H_x \in L^2([0,T]\times\mathbb{R}, \mu(1-\mu)dxdt)$ such that μ satisfies (1.3) weakly, then $I_0(\mu)$ is given by (1.5).

Recall, a function $\mathcal{I} : \mathcal{X} \to [0,\infty]$ on a complete, separable metric space $\mathcal{X}$ is a rate function if it has has closed level sets $\{x : \mathcal{I}(x) \le a\}$. It is a *good* rate function if the level sets are also compact. Also, a sequence $\{X_n\}$ of random variables with values in $\mathcal{X}$ satisfies a large deviation principle (LDP) with speed n and rate function $\mathcal{I}$ if for every Borel set $U \in \mathcal{B}_\mathcal{X}$,

$$\begin{aligned} -\inf_{x\in U^\circ}\mathcal{I}(x) \ &\le \ \liminf_{n\to\infty}\frac{1}{n}\log\Pr(X_n\in U) \\ &\le \ \limsup_{n\to\infty}\frac{1}{n}\log\Pr(X_n\in U) \ \le \ -\inf_{x\in\bar U}\mathcal{I}(x) \end{aligned}$$

where U° is the interior of U and $\bar U$ is the closure of U.

Let $\mathcal{A} = \mathcal{A}(\gamma)$ be the space of all densities μ such that $I_\gamma(\mu) < \infty$ which can be approximated in $D([0,T];M_1)$ by a sequence of smooth densities $\{\mu^n\}$ satisfying (1.3) with respect to an $H^n \in C_K^{1,2}([0,T]\times\mathbb{R})$, and also $I_\gamma(\mu^n) \to I_\gamma(\mu)$.

For general local equilibrium measures (LEM) and deterministic initial configurations (DIC), only a weak large deviation principle is available. The next proposition follows straightforwardly from the methods of [16] (see also [14][Chapter 10]), and replacement estimates in [19], namely Theorem 6.1 and Claims 1,2 [19][Section 6].

Proposition 1.2. *With respect to initial local equilibrium measures (LEM) or deterministic configurations (DIC), corresponding to profile γ, I_γ is a good rate function, and for $U \subset D([0,T];M_1)$,*

$$\begin{aligned} -\inf_{\mu\in\bar{U}} I_\gamma(\mu) \ &\geq\ \limsup_{N\uparrow\infty}\frac{1}{N}\log P\big[\mu^N\in U\big] \\ &\geq\ \liminf_{N\uparrow\infty}\frac{1}{N}\log P\big[\mu^N\in U\big]\ \geq\ -\inf_{\mu\in U^\circ\cap\mathcal{A}} I_\gamma(\mu). \end{aligned}$$

However, with respect to the profiles considered, there is no restriction in the lower bound above.

Proposition 1.3. *With respect to profiles γ associated to local equilibrium measures (LEM) and deterministic configurations (DIC),*

$$\mathcal{A}(\gamma)\ \supset\ \{\mu : I_\gamma(\mu)<\infty\}\,.$$

Corollary 1.4. *With respect to initial local equilibrium mesures (LEM) and deterministic configurations (DIC), the LDP with speed N holds for $\{\mu^N\}$ with good rate function I_γ.*

We note Proposition 1.3, for continuous profiles $\gamma\in M_1(\rho,\rho)$ with $0<\rho<1$ and $0<\gamma(\cdot)<1$ corresponding to local equilibrium measures, was proved in [17], and the associated LDP in Corollary 1.4 with respect to these initial measures is Theorems 3.2, 3.3 [17]. In section 5, we prove Proposition 1.3 through some generalizations of the arguments in [17].

To describe the tagged particle rate function in terms of I_γ, it will be convenient to rewrite (1.3) in terms of a macroscopic 'current' or 'flux' J: That is, define J so that weakly,

$$\partial_x J + \partial_t\mu\ =\ 0;\quad J\ =\ -\frac{1}{2}\partial_x\mu + H_x\mu(1-\mu).$$

Define the function $\mathbb{I}:\mathbb{R}\to[0,\infty]$ as

$$\mathbb{I}(a)\ =\ \inf\Big\{I_\gamma(\mu);\int_0^T J(0,t)dt = \int_0^a \mu_T(x)dx\Big\}.$$

We now give some properties of $\mathbb{I}$.

Proposition 1.5. *With respect to profiles γ corresponding to (LEM) and (DIC) initial measures, $\mathbb{I}$ is finite on $\mathbb{R}$, $\lim_{|a|\uparrow\infty}\mathbb{I}(a)=\infty$, and $\mathbb{I}$ is a good rate function. Further, $\mathbb{I}$ has a unique zero at the LLN constant u_T.*

Moreover, when $I_0(\mu)<\infty$, $\lim_{L\to\infty}\int_0^L \mu_T(x)-\mu_0(x)dx$ converges, and

$$\int_0^T J(0,t)dt\ =\ \lim_{L\to\infty}\int_0^L \mu_T(x)-\mu_0(x)dx. \tag{1.6}$$

Hence, alternatively,

$$\mathbb{I}(a)\ =\ \inf\Big\{I_\gamma(\mu):\lim_{L\to\infty}\int_0^L \mu_T(x)-\mu_0(x)dx = \int_0^a \mu_T(x)dx\Big\}.$$

Equation (1.6) indicates that $\mathbb{I}$ can be written completely in terms of densities μ. This is in some sense a consequence of the enforced ordering of particles in the nearest-neighbor $d = 1$ setting. In contrast, the large deviation rate function in [22] involves an auxilliary current in its description.

We now state the tagged particle large deviation principle.

Theorem 1.6. *Starting under local equilibrium measures (LEM), or deterministic initial configurations (DIC), the scaled positions* $\{X_{N^2T}/N\}$ *satisfy an LDP in scale* N *with rate function* $\mathbb{I}$.

We comment briefly about non-degenerate profiles when $\gamma(x) = 1$ on a half line. In this case, diffusive scaling may not always capture nontrivial LLN's as in (1.1) or large deviations as in Theorem 1.6. For instance, starting under $\xi^{\gamma,N}$ where $\gamma(x) = 1_{(-\infty,0]}(x)$ is the step profile, in Arratia [1] it is shown that $t^{-1/2}x(t) - \sqrt{\log(t)} \to 0$ a.s. which shows that the tagged particle diverges at rate $\sqrt{t\log(t)}$. With respect to large deviations, it is clear the tagged particle, initially at the origin, cannot travel to negative locations. Also, for $a \geq 0$, the condition in $\mathbb{I}(a)$ reduces to $\int_a^\infty \mu_T(x)dx = 0$ which, given that $\mu(t,x)$ satisfies (1.3), is impossible since the density formally becomes positive on $\mathbb{R}$ as soon as $t > 0$. Hence, starting from this step profile configuration, formally $\mathbb{I} = \infty$.

On the other hand, when the profile $\gamma(x) = 1_{[-1,1]}(x)$, a different large deviation behavior is suggested in Theorem 1.9. It would be of interest to further investigate the large deviation behaviors in the degenerate profile setting.

A natural question at this point is to calculate the rate function $\mathbb{I}$. But, this seems difficult in the general case when the zero $u_T \neq 0$, and is not pursued here. Although, we remark for the initial conditions (LEM) and (DIC), for large $|a|$, the proof of the exponential tightness estimate Lemma 3.2 gives non-optimal lower bounds $\mathbb{I}(a) \geq C|a|$, and the proof of Proposition 2.1 gives upper bounds $\mathbb{I}(a) \leq C|a|^3$.

However, some more precise evaluations are available when starting from deterministic configurations corresponding to $\gamma(x) \equiv \rho$ for which the zero $u_T = 0$. Denote $\mathbb{I}(a,\rho,T) = \mathbb{I}(a)$ as the rate for the tagged particle, in diffusive scale, to deviate to a in time T under this initial condition $\xi^{\gamma,N}$.

The next result gives that the order of the rate $\mathbb{I}(a,\rho,T)$ differs according to whether a is small or large, a sort of phase transition.

Theorem 1.7. *Let* $\rho \in (0,1)$. *For all* $a \in \mathbb{R}$, *we have, with respect to constants* c_0, c_1 *depending on* ρ, *that*

$$c_0 \max\left\{\frac{a^2}{\sqrt{T}}, \frac{|a|^3}{T}\right\} \;\leq\; \mathbb{I}(a,\rho,T) \;\leq\; c_1 \max\left\{\frac{a^2}{\sqrt{T}}, \frac{|a|^3}{T}\right\}.$$

In the proof, bounds on the constants can be recovered, for instance $c_0 \geq \min\{\rho^2/(\sqrt{6}), 2\rho^3/3\}$, and a more messy bound for c_1 is also clear.

The asymptotics $\mathbb{I}(a,\rho,T) \sim O(|a|^2/\sqrt{T})$ for small $|a|$ recalls familiar Gaussian expansions, however, the large $|a|$ growth seem intriguing, perhaps connected with totally asymmetric nearest-neighbor exclusion (TASEP) effects. That is, for the tagged particle to deviate to aN far from the origin, it must drive $O(|a|N)$ particles away, so that perhaps the process behaves like a driven system like TASEP. Of course, when $\rho = 0$, only the tagged particle is present in the system, and then $\mathbb{I}(a,\rho,T) \equiv 0$ given the diffusive scaling.

We remark on these last points that in Derrida-Gerschenfeld [10], starting from a local equilibrium measure with step profile $\gamma(x) = \rho_l 1_{(-\infty,0]} + \rho_r 1_{(0,\infty)}$, the pressure of the current $J_{0,1}(t)$ across the bond $(0,1)$, $\lim_{t\uparrow\infty} t^{-1/2}\log E[\exp\{\lambda J_{0,1}(t)\}] = F(\rho_l,\rho_r,\lambda)$, is found from which, interestingly, non-Gaussian fluctuations are determined. Also, formal asymptotics with F give $P(J_{0,1}(t) = a) \sim \exp[\sqrt{t}\{-\frac{\pi^2}{12}a^3 + \cdots\}]$, for large t and a. Given relations between the tagged particle position $x(t)$ and the current, e.g. those in subsection 3.1, formally one might understand the third order bounds in Theorem 1.7 from these current expansions. In fact, by our methods, to be reported upon later, one can prove $\lim_t t^{-1/2}\log P(J_{0,1}(t) \geq a)$ is on order a^3 for large a.

Also, we note, in Sasamoto Section 5.2 [28], fluctuations in the 'KPZ' class are discussed for tagged particles in TASEP starting from an 'alternating' initial condition (every other vertex is occupied), and other initial conditions. In particular, the scaling limits are of 'Airy' process type whose marginal distribution functions $F(x)$ decay (cf. Baik-Buckingham-DiFranco [2]), with respect certain powers θ_0, θ_1 and constants c_0, c_1, on order $|x|^{\theta_0}e^{-c_0|x|^3}$ as $x \downarrow -\infty$, and $1 - |x|^{\theta_1}e^{-c_1 x^{3/2}}$ as $x \uparrow \infty$. Formally, one is tempted to link the cubic order $\mathbb{I}(a,\rho,T) \sim O(|a|^3)$ for large $|a|$ in terms of the Airy process exponents. It would be interesting to investigate further such connections.

We now refine the behavior of $\mathbb{I}(a,\rho,T)$ near the zero $a = 0$. Arratia's CLT variance σ^2, mentioned earlier, can be computed by adding static and dynamic contributions, due to initial configuration and later motion fluctuations respectively. However, starting from deterministic initial configurations, only the dynamical contribution would be present, and we show later in Proposition 4.5 that this part of Arratia's variance is $\sigma^2_{dyn} = (1-\rho)/\rho\sqrt{\pi}$.

Theorem 1.8. *For $\rho \in (0,1)$, we have*

$$\lim_{|a|\downarrow 0} \frac{1}{a^2}\mathbb{I}(a,\rho,T) \;=\; \frac{1}{2\sigma^2_{dyn}\sqrt{T}} \;=\; \frac{\rho}{1-\rho}\frac{\sqrt{\pi}}{2\sqrt{T}}.$$

Finally, as a contrast to the results in Theorem 1.7, we show quadratic large deviation upper bounds starting from the non-degenerate configuration $\xi^{\gamma_1,N}$ where $\xi^{\gamma_1,N}(x) = 1$ for $|x| \leq N$ and $\xi^{\gamma_1,N}(x) = 0$ otherwise. Here, $\gamma_1(x) = 1_{[-1,1]}(x)$. Note the associated LLN speed $u_T = 0$.

Theorem 1.9. *Starting under $\xi^{\gamma_1,N}$, there exists $c_2 > 0$ such that, for $a \geq 0$,*

$$\limsup_{N\uparrow\infty} \frac{1}{N}\log P\left(|X(N^2T)/N| \;\geq a\right) \;\leq\; -c_2 a^2.$$

The interpretation is that although the tagged particle in the configurations $\xi^{\gamma_1,N}$ is trapped in the middle of a large segment of particles, to displace large distances, as there are only $O(N)$ number or particles in the system, the cost is not as great as under $\xi^{\rho,N}$ where there are an infinite number of particles. At the same time, there is a positive density of particles to the left and right of the origin, unlike for the profile $\gamma(x) = 1_{(-\infty,0]}(x)$, which slows down the tagged particle so that deviations to $a \in \mathbb{R}$ have finite cost in diffusive scale.

Rough estimates for c_2 can be found from the proof.

The plan of the paper is now to develop preliminary estimates in section 2. In section 3, we prove Proposition 1.5 and Theorem 1.6. Then, in section 4, we prove

Theorems 1.7, 1.8. and 1.9. These last two sections can be read independently of each other. Finally, in section 5, as remarked earlier, we prove Proposition 1.3, and other approximations.

2. Preliminary estimates

In several subsections, we prove $\mathbb{I}$ is finite and lower semi-continuous, and give approximation, entropy and current bounds for densities with finite rate.

2.1. **Finiteness of $\mathbb{I}$.** Recall from the discussion near (1.3) that $I_\gamma(\mu)$ has explicit representation, and $\sigma_t(x) = (2\pi t)^{-1/2}\exp\{-x^2/2t\}$. Consider a C^∞ smooth 'bump' function, supported on $[-1,1]$, to be specific, say

$$\psi_0(x) \;=\; \exp\big\{-1/(1-x^2)\big\},$$

and define the smooth, anti-symmetric function,

$$\psi(x) \;=\; \begin{cases} -\psi_0(2(x+1/2)) & \text{for } x\le 0\\ \;\;\psi_0(2(x-1/2)) & \text{for } x\ge 0.\end{cases}$$

Define also the anti-derivative $\Psi(x) = \int_{-1}^x \psi(y)dy$, supported on $[-1,1]$.

Let $\gamma \in M_1(\rho_*,\rho^*)$ be a profile associated to an initial (LEM) local equilibrium measure or a (DIC) deterministic configuration. Recall u_T is the LLN speed associated to γ (cf. (1.1)). We now show that $\mathbb{I}$ is finite on $\mathbb{R}$.

Proposition 2.1. *For $\mathfrak{c}\in\mathbb{R}$, $\mathbb{I}(\mathfrak{c})<\infty$. Moreover, on any interval $[a,b]\subset\mathbb{R}$, $\sup_{\mathfrak{c}\in[a,b]}\mathbb{I}(c)<\infty$.*

Proof. Consider

$$\mu(s,x) \;=\; \sigma_s*\gamma(x) + (\lambda\epsilon(s/T))\psi(x/L)$$

where $\epsilon(t)$ is a smooth, increasing function which vanishes for $0\le t\le 1/10$, and $\epsilon(1)=1$, and $L\neq 0$. At time $s=T/10$, $0<\gamma_*<\sigma_s*\gamma<\gamma^*<1$ for some constants γ_*,γ^*. We will take $0\le\lambda<\min\{\gamma_*,1-\gamma^*\}/2$, small enough so that $\gamma_*/2\le\mu\le(1-\gamma^*)/2$ for $T/10\le t\le T$.

Then, as μ follows the heat equation for $[0,T/10]$, μ satisfies (1.3) with respect to H_x, supported on $[T/10,T]\times[-|L|,|L|]$ given by

$$H_x \;=\; \begin{cases} \frac{1}{\mu(1-\mu)}\Big[\frac{\lambda\epsilon(s/T)}{2L}\psi'\big(\frac{x}{L}\big) - \frac{\lambda L\epsilon'(s/T)}{T}\Psi\big(\frac{x}{L}\big)\Big] & \text{for } \frac{T}{10}\le s\le T, |x|\le|L|\\ 0 & \text{otherwise .}\end{cases}$$

Also, as $\mu_0=\gamma$, we have $h(\mu_0;\gamma)=0$, and

$$\begin{aligned} I_0(\mu) &= \frac{1}{2}\int_{T/10}^T\int \frac{1}{\mu(1-\mu)}\Big[\frac{\lambda\epsilon(s/T)}{2L}\psi'\Big(\frac{x}{L}\Big) - \frac{\lambda L\epsilon'(s/T)}{T}\Psi\Big(\frac{x}{L}\Big)\Big]^2 dxds\\ &\le \frac{4\epsilon^*}{\gamma_*(1-\gamma^*)}\Big[\frac{\lambda^2 T}{4|L|}\int_{-1}^1\psi'(x)^2dx + \frac{\lambda^2|L|^3}{T}\int_{-1}^1\Psi(x)^2dx\Big] \end{aligned} \tag{2.1}$$

where $\epsilon^* = 1+\|\epsilon'\|^2_{L^\infty}$. Compute now

$$\begin{aligned}\int_0^\infty\big[\mu_T(x)-\mu_0(x)\big]dx &= \lambda L\int_0^1\psi(x)dx + \int_0^{u_T}\sigma_T*\gamma(x)dx, \quad\text{and}\\ \int_0^{\mathfrak{c}}\mu_T(x)dx &= \int_0^{\mathfrak{c}}\sigma_T*\gamma(x)dx + \lambda L\int_0^{|\mathfrak{c}|/|L|}\psi(x)dx.\end{aligned}$$

Then, the restriction, $\int_0^T J(0,t)dt = \int_0^\infty \mu_T(x) - \mu_0(x)dx = \int_0^{\mathfrak{c}} \mu_T(x)dx$, holds when

$$\lambda L \int_{|\mathfrak{c}|/|L|}^1 \psi(x)dx \ = \ \int_{u_T}^{\mathfrak{c}} \sigma_T * \gamma(x)dx.$$

If $\mathfrak{c} = u_T$, we may take $\lambda = 0$, and so $\mathbb{I}(u_T) = 0$. For $\mathfrak{c} \neq u_T$, let $\lambda > 0$, and note that the left-side vanishes for $|L| \le |\mathfrak{c}|$ and diverges to $\pm\infty$ as $L \to \pm\infty$. Hence, a proper choice of L allows to verify $\mathbb{I}(\mathfrak{c}) < \infty$.

In particular, we can see, by varying L, with respect to any finite interval $[a,b]$, $\sup_{\mathfrak{c}\in[a,b]} \mathbb{I}(\mathfrak{c}) < \infty$. □

2.2. **Approximation and limit estimates.** We state an approximation result whose proof follows the method indicated in Landim [17]. In addition, we show a certain regularity at infinity. Proofs of these results are given in Section 5.

Proposition 2.2. *Let μ be a density such that $I_0(\mu) < \infty$. Then, there is a sequence $\mu^n \in D([0,T]; M_1)$ such that*

(i) *for $0 < \delta_n < 1$, $\delta_n \le \mu^n(t,x) \le 1 - \delta_n$ for $(t,x) \in [0,T] \times \mathbb{R}$,*
(ii) *$\mu^n \in C^\infty([0,T] \times \mathbb{R})$,*
(iii) *$H^n_x \in C^\infty([0,T] \times \mathbb{R}) \cap L^\infty([0,T] \times \mathbb{R})$,*
(iv) *$\|\partial_x^{(k)}\partial_t^{(l)}\mu^n\|_{L^\infty([0,T]\times\mathbb{R})} < \infty$ for $k,l \ge 1$,*
(v) *$\mu^n \to \mu$ in $D([0,T]; M_1)$, and*
(vi) *$I_0(\mu^n) \to I_0(\mu)$.*
(vii) *Also, suppose $\hat\gamma \in M_1(\rho_*, \rho^*)$ is a.s. continuous, and $0 < \hat\gamma(x) < 1$ for $x \in \mathbb{R}$. Then, if $h(\mu_0; \hat\gamma) < \infty$, we have $h(\mu^n_0; \hat\gamma) \to h(\mu_0; \hat\gamma)$.*
(viii) *In addition, if $\mu_0(x) \equiv \rho$, then μ^n_0 can be taken as $\mu^n_0(x) \equiv \rho$.*

We now turn to the limits at infinity.

Lemma 2.3. *Let $\hat\gamma \in M_1(\rho_*, \rho^*)$, and let μ be a smooth density satisfying (1.3) such that $\|\partial_x\mu\|_{L^\infty([0,T]\times\mathbb{R})} < \infty$, $h(\mu_0; \hat\gamma) < \infty$ and $I_0(\mu) < \infty$. Then, we have*

$$\lim_{|y|\uparrow\infty} \sup_{t\in[0,T]} |\mu(t,y) - \hat\gamma(y)| \ = \ 0.$$

The next lemma will be used in the proof of Theorem 1.8.

Lemma 2.4. *Let $\{\mu^n\}$ be a sequence of smooth densities satisfying (1.3) where $\mu^n_0(x) \equiv \rho$, $\|\partial_x\mu^n\|_{L^2([0,T]\times\mathbb{R})} \downarrow 0$, and $I_0(\mu^n) \downarrow 0$. Then,*

$$\|\mu^n - \rho\|_\infty := \sup_{\substack{x\in\mathbb{R}\\ t\in[0,T]}} |\mu^n(t,x) - \rho| \ \to \ 0 \quad \text{ as } n \uparrow \infty.$$

2.3. **Entropy and current estimates.** We give estimates on the entropy and current with respect to a density μ such that $I_\gamma(\mu) < \infty$.

Lemma 2.5. *Let μ be a density with finite rate $I_0(\mu) < \infty$. Suppose for $\gamma \in M_1(\rho_*, \rho^*)$ with $0 < \rho_*, \rho^* < 1$, that $h(\mu_0; \gamma) < \infty$. Then,*

$$\int_0^T \int (\partial_x\mu_s)^2 dxds \ < \ \infty \ \text{ and } \ \int_0^T \int J^2(x,t)dxdt \ < \infty.$$

Proof. By Proposition 2.2, we can approximate μ by a smooth density μ^n such that $\delta_n < \mu^n < 1 - \delta_n$. Let $\hat\gamma \in M_1(\rho_*, \rho^*)$ be a smooth function such that $0 < \gamma_* < \hat\gamma < \gamma^* < 1$ for some constants γ_*, γ^*, and $h(\gamma; \hat\gamma) < \infty$. Let also G_L be a smooth, nonnegative, compactly supported function in $[-L, L]$, bounded by 1, which equals 1 on $[-L+1, L-1]$, and $\sup_L \int_{A_L} (G_L')^2/G_L dx < \infty$ where $A_L = [L-1, L] \cup [-L, -L+1]$. In the following, the constant C may change line to line. Then,

$$\begin{aligned}
&\partial_t \int G_L(x) h_d(\mu_t^n(x); \hat\gamma) dx \qquad (2.2)\\
&= \int G_L \partial_t \mu_t^n \log \frac{\mu_t^n}{1-\mu_t^n} \frac{1-\hat\gamma}{\hat\gamma} dx\\
&= \int G_L \Big[(1/2)\partial_{xx}\mu_t^n - \partial_x (H_x^n \mu_t^n (1-\mu_t^n))\Big] \log \frac{\mu_t^n}{1-\mu_t^n} \frac{1-\hat\gamma}{\hat\gamma} dx\\
&= \int \Big[-(1/2)\partial_x \mu_t^n + H_x^n (\mu_t^n(1-\mu_t^n))\Big] \partial_x \Big[G_L \log \frac{\mu_t^n}{1-\mu_t^n} \frac{1-\hat\gamma}{\hat\gamma}\Big] dx\\
&= -\frac{1}{2} \int G_L(x) \frac{(\partial_x \mu_t^n)^2}{\mu_t^n(1-\mu_t^n)} dx + \int G_L(x) H_x^n \partial_x \mu_t^n dx\\
&\quad + \frac{1}{2} \int G_L(x) \frac{\partial_x \hat\gamma \partial_x \mu^n}{\hat\gamma(1-\hat\gamma)} dx - \int G_L(x) H_x^n \mu^n (1-\mu^n) \frac{\partial_x \hat\gamma}{\hat\gamma(1-\hat\gamma)} dx\\
&\quad + \int_{A_L} G_L'(x) \Big[-(1/2)\partial_x \mu_t^n + H_x^n(\mu_t^n(1-\mu_t^n))\Big] \log \frac{\mu_t^n}{1-\mu_t^n} \frac{1-\hat\gamma}{\hat\gamma} dx.
\end{aligned}$$

Hence, by Schwarz inequality and $0 \le \mu^n \le 1$, we can bound, with respect to a constant C independent of L,

$$\begin{aligned}
&\int G_L(x) h_d(\mu_T^n(x); \hat\gamma(x)) dx + \frac{1}{4} \int_0^T \int G_L(x) \frac{(\partial_x \mu_s^n)^2}{\mu_s^n(1-\mu_s^n)} dx ds\\
&\le \int G_L(x) h_d(\mu_0^n(x); \hat\gamma(x)) dx + C \int_0^T \int (H_x^n)^2 \mu_s^n (1-\mu_s^n) dx ds\\
&\quad + CT \int G_L(x) \frac{(\partial_x \hat\gamma)^2}{\hat\gamma^2(1-\hat\gamma)^2} dx + C \int_{A_L} [(G_L')^2/G_L][\log \frac{\mu_t^n}{1-\mu_t^n} \frac{1-\hat\gamma}{\hat\gamma}]^2 dx.
\end{aligned}$$

We can take $L \uparrow \infty$, so that the last term vanishes by Lemma 2.3. Then, as $h_d(\alpha; \beta) \ge 0$, by monotone convergence,

$$\begin{aligned}
&h(\mu_T^n; \hat\gamma) + \frac{1}{4} \int_0^T \int \frac{(\partial_x \mu_s^n)^2}{\mu_s^n(1-\mu_s^n)} dx ds\\
&\quad \le h(\mu_0^n; \hat\gamma) + C \int_0^T \int (H_x^n)^2 \mu_s^n (1-\mu_s^n) dx ds + CT \|\partial_x \hat\gamma\|_{L^2}.
\end{aligned}$$

Since, $I_0(\mu^n) \to I_0(\mu) < \infty$, and $h(\mu_0^n; \hat\gamma) \to h(\mu_0; \hat\gamma) < \infty$, the right side above is uniformly bounded as $n \uparrow \infty$. Hence, as $\mu_s^n(1-\mu_s^n) \le 1/4$, we have

$$\begin{aligned}
\int_0^T \int (\partial_x \mu_s^n)^2 dx ds &\le h(\mu_0^n; \hat\gamma) + C \int_0^T \int (H_x^n)^2 \mu_s^n (1-\mu_s^n) dx ds + CT \|\partial_x \hat\gamma\|_{L^2}\\
&\le h(\mu_0^n; \hat\gamma) + C I_0(\mu^n) + C \|\partial_x \hat\gamma\|_{L^2}
\end{aligned}$$

is uniformly bounded also.

We may now extract a subsequence $\partial_x \mu^{n_k}$ converging weakly in L^2 to ζ. We will drop the subscript k in the following. Since $\mu^n \to \mu$ in $D([0,T]; M_1)$, we have for smooth compactly supported G that $\int G \partial_x \mu^n dx ds = \int -G_x \mu^n dx ds$ converges to both $\int G\zeta dx ds$ and $\int -G_x \mu dx ds$. Hence, $\partial_x \mu$ exists weakly in L^2, and $\partial_x \mu = \zeta$.

In particular, noting Skorohod convergence gives at the endpoint $t = T$ that $\mu^n_T \to \mu_T$, and that $\partial_x\mu^n \to \partial_x\mu$ weakly in L^2, by lower semi-continuity of the entropy and L^2 norm, we recover the desired entropy and L^2 bounds on μ_T and $\partial_x\mu$ respectively. Also, noting $0 \le \mu \le 1$ and $I_0(\mu) < \infty$, we have $H_x\mu(1-\mu) \in L^2([0,T]\times\mathbb{R})$. Hence, $J = -1/2\partial_x\mu + H_x\mu(1-\mu)$ belongs to $L^2([0,T]\times\mathbb{R})$. □

The proof gives the following corollaries, useful in the proofs of Theorems 1.7 and 1.8. The first is a formula for the rate function $I_0(\mu)$.

Corollary 2.6. *Let μ be a smooth density, strictly bounded between* 0 *and* 1*, such that $\|\partial_x\mu\|_{L^\infty} < \infty$, $I_0(\mu) < \infty$ and $h(\mu_0;\hat\gamma) < \infty$ with respect to a smooth $\hat\gamma \in M_1(\rho_*,\rho^*)$ also strictly bounded between* 0 *and* 1*. Then,*

$$\begin{aligned} I_0(\mu) &= \frac{1}{8}\int_0^T\int \frac{(\partial_x\mu)^2}{\mu(1-\mu)}dxdt + \frac{1}{2}\left[h(\mu_T;\hat\gamma) - h(\mu_0;\hat\gamma)\right] \\ &\quad + \frac{1}{2}\int_0^T\int \frac{J\partial_x\hat\gamma}{\hat\gamma(1-\hat\gamma)}dxdt + \frac{1}{2}\int_0^T\int \frac{J^2}{\mu(1-\mu)}dxdt. \end{aligned}$$

Proof. From the proof of Lemma 2.5, applied to μ in place of μ^n, and noting $J = -(1/2)\partial_x\mu + H_x\mu(1-\mu)$, we have

$$\int_0^T\int \frac{(\partial_x\mu)^2}{\mu(1-\mu)}dxdt < \infty \quad \text{and} \quad \int_0^T\int \frac{J^2}{\mu(1-\mu)}dxdt < \infty.$$

Hence, integrating (2.2) and taking limit on L, we obtain

$$h(\mu_T;\hat\gamma) - h(\mu_0;\hat\gamma) = \int_0^T\int \frac{J\partial_x\mu}{\mu(1-\mu)}dxdt - \int_0^T\int \frac{J\partial_x\hat\gamma}{\hat\gamma(1-\hat\gamma)}dxdt.$$

Then,

$$\begin{aligned} I_0(\mu) &= \frac{1}{2}\int_0^T\int H_x^2\mu(1-\mu)dxdt \\ &= \frac{1}{2}\int_0^T\int \frac{[(1/2)\partial_x\mu + J]^2}{\mu(1-\mu)}dxdt \\ &= \frac{1}{8}\int_0^T\int \frac{(\partial_x\mu)^2}{\mu(1-\mu)}dxdt \\ &\quad + \frac{1}{2}\int_0^T\int \frac{J\partial_x\mu}{\mu(1-\mu)}dxdt + \frac{1}{2}\int_0^T\int \frac{J^2}{\mu(1-\mu)}dxdt. \end{aligned}$$

By substituting for the middle term, we recover the desired formula for $I_0(\mu)$. □

Next, we give a concrete bound on $\|\partial_x\mu\|_{L^2}$.

Corollary 2.7. *Let μ be a smooth density, strictly bounded between* 0 *and* 1*, such that $\|\partial_x\mu\|_{L^\infty} < \infty$, $I_0(\mu) < \infty$ and $h(\mu_0;\hat\gamma) < \infty$ with respect to a smooth $\hat\gamma \in M_1(\rho_*,\rho^*)$ also strictly bounded between* 0 *and* 1*. Then,*

$$\frac{1}{6}\|\partial_x\mu\|_{L^2}^2 \le h(\mu_0;\hat\gamma) + I_0(\mu) + T\|\partial_x\hat\gamma/(\hat\gamma(1-\hat\gamma))\|_{L^2}^2. \tag{2.3}$$

Proof. Since $\mu(1-\mu) \le 1/4$, from Corollary 2.6 and Schwarz inequality and $ab \le (a^2+b^2)/2$, we obtain

$$\|\partial_x\mu\|_{L^2}^2 \le h(\mu_0;\hat\gamma) + \frac{1}{2}\|J\|_{L^2}^2 + \frac{T}{2}\|\partial_x\hat\gamma/(\hat\gamma(1-\hat\gamma))\|_{L^2}^2 + 2I_0(\mu).$$

As $J = -(1/2)\partial_x\mu + H_x\mu(1-\mu)$, and $\|H_x\mu(1-\mu)\|^2_{L^2} \leq I_0(\mu)$, we obtain

$$\|\partial_x\mu\|^2_{L^2} \quad \leq \quad h(\mu_0;\hat{\gamma}) + \frac{2}{8}\|\partial_x\mu\|^2_{L^2} + \frac{2}{4}I_0(\mu) + 2I_0(\mu) + T\|\partial_x\hat{\gamma}/(\hat{\gamma}(1-\hat{\gamma}))\|^2_{L^2}$$

from which the result follows. □

Corollary 2.8. *Let $\{\mu^n\}$ be a sequence of densities such that $\lim_{n\to\infty} I_0(\mu^n) = 0$, and for each n, $\mu^n_0(x) \equiv \rho$, $\|\partial_x\mu^n\|_{L^\infty} < \infty$, and μ^n is strictly bounded between 0 and 1. Then,*

$$\lim_{n\to\infty} \|\partial_x\mu^n\|_{L^2([0,T]\times\mathbb{R})} \;=\; 0.$$

Proof. The proof follows by taking $\hat{\gamma}(x) \equiv \rho$ in Corollary 2.7. □

2.4. **Current-mass relation.** We give some estimates on the current J, and prove at the end the current-mass relation (1.6).

Lemma 2.9. *Let μ be a density such that $I_0(\mu) < \infty$. Let $\{\mu^n\}$ be a sequence converging to μ with properties (i)-(viii) in Proposition 2.2. Then, for each $x \in \mathbb{R}$,*

$$\lim_{n\to\infty}\int_0^T J^n(x,t)dt \;=\; \int_0^T J(x,t)dt.$$

Further, $\int_0^T J(x,t)dt$ is Lipschitz as a function of x, and

$$\lim_{|x|\uparrow\infty}\int_0^T J(x,t)dt \;=\; 0.$$

Proof. First, $\partial_t\mu^n + \partial_x J^n = 0$, and

$$J^n \;=\; (-1/2)\partial_x\mu^n + H^n_x\mu^n(1-\mu^n). \tag{2.4}$$

From Lemma 2.5, as $\partial_x\mu^n$, and $H^n_x\mu^n(1-\mu^n)$ are uniformly bounded in L^2, we have also $\sup_n \|J^n\|_{L^2([0,T]\times\mathbb{R})} < \infty$, and we can find a subsequence where $J^{n_k} \to \phi$ and $\partial_x\mu^{n_k} \to \partial_x\mu$ weakly in L^2. For simplicity, we drop again the subscript k in the following. Note

$$\begin{aligned}
\int_0^T\int J^n G_x dxdt &= -\int_0^T\int J^n_x G dxdt \\
&= \int_0^T\int \partial_t\mu^n G dxdt \\
&= \int G(T,x)\mu^n_T dx - \int G(0,x)\mu^n_0 dx - \int_0^T\int \mu^n G_t dxdt \\
&\to \int G(T,x)\mu_T dx - \int G(0,x)\mu_0 dx - \int_0^T\int \mu G_t dxdt
\end{aligned}$$

and also

$$\int_0^T\int J^n G_x dxdt \;\to\; \int_0^T\int \phi G_x dxdt.$$

Hence, $\phi_x = -\partial_t\mu$ weakly in L^2. Then, $\phi_x = (-1/2)\partial_{xx}\mu + \partial_x[H_x\mu(1-\mu)]$ weakly in L^2, and so $\phi = (-1/2)\partial_x\mu + H_x\mu(1-\mu) + C(t)$ with respect to a function $C(t)$ not dependent on x. But, $\phi, \partial_x\mu, H_x\mu(1-\mu)$ belong to $L^2([0,T]\times\mathbb{R})$, hence $C(t) \equiv 0$. In particular, $\phi = J$, and the whole sequence $J^n \to J$ weakly in L^2.

We now utilize that the difference of the currents across a and b up to time T is equal to the difference in the masses in the interval $[a,b]$ from times T to 0, that is by integrating (2.4):

$$\int_0^T J^n(t,a)dt - \int_0^T J^n(t,b)dt \;=\; \int_a^b \mu_T^n(x) - \mu_T^n(0)dx.$$

Hence $|\int_0^T J^n(t,a)dt - \int_0^T J^n(t,b)dt| \leq |b-a|$ as $0 \leq \mu^n \leq 1$. In particular, $\int_0^T J^n(t,a)dt$ is Lipschitz in a. Moreover, a subsequence, $\int_0^T J^{n_k}(t,\cdot)dt \to \psi(\cdot)$ converges uniformly on compact subsets to a Lipschitz function ψ. As $J^n \to J$ weakly, for $G \in L^2(\mathbb{R})$, we conclude by a limit argument that $\int G(a)\int_0^T J(t,a)dtda = \int G(a)\psi(a)da$ and so $\psi(a) = \int_0^T J(t,a)dt$. In particular, the whole sequence $\int_0^T J^n(t,\cdot)dt \to \int_0^T J(t,\cdot)dt$, and the limit $\int_0^T J(t,\cdot)dt$ is Lipschitz.

Finally, since

$$\int \Big[\int_0^T J(t,x)dt\Big]^2 dx \;\leq\; T\int\int_0^T J^2(t,x)dtdx \;<\; \infty,$$

we have, for $\kappa \neq 0$, that $\int_{n\kappa}^{(n+1)\kappa}\int_0^T J(t,x)dtdx \to 0$. Then, as $\int_0^T J(t,x)dt$ is Lipschitz in x, we obtain the pointwise limit $\int_0^T J(x,t)dt \to 0$ as $|x| \uparrow \infty$. □

Finally, we prove the 'current-mass' relation (1.6) in the second part of Proposition 1.5.

Lemma 2.10. *For μ be a density such that $I_0(\mu) < \infty$, we have*

$$\int_0^\infty \mu_T(x) - \mu_0(x)dx \;:=\; \lim_{L\uparrow\infty}\int_0^L \big[\mu_T(x) - \mu_0(x)\big]dx \quad \text{converges,}$$

and

$$\int_0^T J(0,t)dt \;=\; \int_0^\infty \mu_T(x) - \mu_0(x)dx.$$

Proof. Write,

$$\int_0^T J(0,t)dt - \int_0^T J(L,t)dt \;=\; \int_0^L \mu_T(x) - \mu_0(x)dx$$

and note that $\lim_{L\to\infty}\int_0^T J(L,t)dt = 0$ to obtain the claims. □

2.5. **Lower semi-continuity of $\mathbb{I}$.** Lower semi-continuity follows from the previous estimates.

Lemma 2.11. $\mathbb{I}$ *is lower semi-continuous.*

Proof. We first consider when starting from a local equilibrium measure and $I_\gamma = I_\gamma^{LE}$. Let $\{a_k\}$ be a convergent sequence $a_k \to a$. From Proposition 2.1, we have $\sup_k \mathbb{I}(a_k) < \infty$. Then, by Proposition 2.2 and Lemma 2.9, we can find smooth densities $\{\mu^k\}$ so that $|I_\gamma^{LE}(\mu^k) - \mathbb{I}(a^k)| < k^{-1}$ and $|\int_0^T J^k(0,t)dt - \int_0^{a_k}\mu_T^k(x)dx| \leq k^{-1}$.

As I_γ^{LE} is a good rate function, a subsequence can be found where μ^k converges to $\hat\mu$ in $D([0,T];M_1)$. By Lemma 2.9, we have $\int_0^T J^k(0,t)dt \to \int_0^T \hat J(0,t)dt$. Also, as $\mu_T^k \to \hat\mu_T$, and $a_k \to a$, we have $\int_0^{a_k}\mu_T^k(x)dx \to \int_0^a \hat\mu_T(x)dx$.

Then, $\int_0^T \hat{J}(0,t)dt = \int_0^a \hat{\mu}_T(x)dx$. Using lower semi-continuity of I_γ^{LE}, the desired lower semi-continuity follows as $\lim_k \mathbb{I}(a^k) = \lim_k I_\gamma^{LE}(\mu^k) \geq I_\gamma^{LE}(\hat{\mu}) \geq \mathbb{I}(a)$.

Starting from a deterministic configuration, we can repeat the steps with I_γ^{LE} replaced by I_0. The measures $\{\mu^k\}$ are such that μ_0^k converges to γ. Hence, the limit $\hat{\mu}$ in addition satisfies $\hat{\mu}_0 = \gamma$ which gives $I_0(\hat{\mu}) = I_\gamma^{DC}(\hat{\mu})$ and so $\mathbb{I}$ is also lower semi-continuous in this case. □

3. Proofs of Proposition 1.5 and Theorem 1.6

The proofs follow in several steps divided into subsections. The first step is to describe some relations between a tagged particle and the current across the bond $(-1, 0)$. Next, we give some estimates leading to the proof of Proposition 1.5. Then, weak upper and lower large deviation bounds are then established, and finally Proposition 1.5 and Theorem 1.6 are proved.

3.1. **Tagged particle, current relations.** For $x \in \mathbb{Z}$ and $t \geq 0$, define $J_{x,x+1}(t)$ as the integrated current up to time t across the bond $(x, x+1)$, that is the number of particles which crossed from x to $x+1$ up to time t minus the number of particles which moved from $x+1$ to x in time t. It is well known (cf. Liggett [20], DeMasi-Ferrari [8]) that, for integers $r > 0$,

$$\big\{X_t \geq r\big\} \;=\; \Big\{J_{-1,0}(t) \geq \sum_{x=0}^{r-1} \eta_t(x)\Big\}. \tag{3.1}$$

Similarly, for $r < 0$,

$$\big\{X_t \leq r\big\} \;=\; \Big\{J_{-1,0}(t) \leq -\sum_{x=r}^{-1} \eta_t(x)\Big\} \tag{3.2}$$

and

$$\big\{X_t \leq 0\big\} \;=\; \big\{J_{-1,0}(t) \leq 0\big\}.$$

Also, from a moment's thought, we have

$$J_{x-1,x}(N^2t) - J_{x,x+1}(N^2t) \;=\; \eta_{N^2t}(x) - \eta_0(x).$$

We would like to make a summation-by-parts,

$$J_{-1,0}(N^2t) \;=\; \sum_{x\geq 0} J_{x-1,x}(N^2t) - J_{x,x+1}(N^2t) \;=\; \sum_{x\geq 0} \eta_{N^2t}(x) - \eta_0(x)$$

to write the current across the bond $(-1, 0)$ in terms of the empirical process. However, the above display is only formal as the sum on the right may not converge. To treat it carefully, we introduce a 'cutoff' function as in Rost-Vares [26]. For $n \geq 1$, let

$$G_n(u) \;=\; 1_{[0,n]}(u)(1 - u/n).$$

Also, denote for a function $G \in C_K^\infty(\mathbb{R})$,

$$Y_t^N(G) \;=\; \frac{1}{N}\sum_x G(x/N)\eta_{N^2t}(x).$$

Then,

$$\begin{aligned} Y_t^N(G_n) - Y_0^N(G_n) &= \frac{1}{N}\sum_x G_n(x/N)\Big(J_{x-1,x}(N^2t) - J_{x,x+1}(N^2t)\Big) \\ &= \frac{1}{N}\sum_x \Big(G_n(x/N) - G_n(x-1/N)\Big)J_{x-1,x}(N^2t) \\ &= \frac{1}{N}J_{-1,0}(N^2t) - \frac{1}{N}\sum_{x=1}^{nN}\frac{1}{nN}J_{x-1,x}(N^2t). \end{aligned}$$

This implies

$$\frac{1}{N}J_{-1,0}(N^2t) \;=\; Y_t^N(G_n) - Y_0^N(G_n) + \frac{1}{N}\sum_{x=1}^{nN}\frac{1}{nN}J_{x-1,x}(N^2t).$$

Hence, for $a > 0$,

$$\begin{aligned} \Big\{X_{N^2t}/N \geq a\Big\} &= \Big\{\frac{1}{N}J_{-1,0}(N^2t) \geq \frac{1}{N}\sum_{x=0}^{\lfloor aN\rfloor}\eta_{N^2t}(x)\Big\} \\ &= \Big\{Y_t^N(G_n) - Y_0^N(G_n) \\ &\qquad + \frac{1}{nN^2}\sum_{x=1}^{nN}J_{x-1,x}(N^2t) \geq \frac{1}{N}\sum_{x=0}^{\lfloor aN\rfloor}\eta_{N^2t}(x)\Big\}. \end{aligned} \tag{3.3}$$

A similar statement holds for $a \leq 0$, namely,

$$\begin{aligned} \Big\{X_{N^2t}/N \leq a\Big\} &= \Big\{Y_t^N(G_n) - Y_0^N(G_n) \\ &\qquad + \frac{1}{nN^2}\sum_{x=1}^{nN}J_{x-1,x}(N^2t) \leq -\frac{1}{N}\sum_{x=\lfloor aN\rfloor}^{-1}\eta_{N^2t}(x)\Big\} \end{aligned}$$

where for $a = 0$ we take $\sum_{x=0}^{-1}\eta_{N^2t}(x) = 0$.

Therefore, heuristically, the tagged particle large deviations should be given in terms of the rate for the empirical density I_γ under a certain restriction, as long as the contribution from the term $(1/nN^2)\sum_{x=1}^{nN}J_{x-1,x}(N^2t)$ is superexponentially small as $n, N \uparrow \infty$.

3.2. **Superexponential estimate.** From (3.3), we see that to relate the tagged particle deviations to those for the empirical density, we need the following estimate.

Proposition 3.1. *For each $\lambda > 0$, starting from local equilibrium measures or deterministic configurations,*

$$\lim_{n\uparrow\infty}\lim_{N\uparrow\infty}\frac{1}{N}\log E\exp\Big|\frac{\lambda N}{nN^2}\sum_{x=1}^{nN}J_{x-1,x}(N^2t)\Big| \;=\; 0.$$

Proof. By the inequality $e^{|x|} \leq e^x + e^{-x}$, we can remove the absolute value in the last display. Now, note that

$$\exp\Big\{\frac{\lambda N}{nN^2}\sum_{x=1}^{nN} J_{x-1,x}(N^2t) - \sum_{x=1}^{nN}(e^{\lambda/nN}-1)\int_0^{N^2t}\eta_{x-1}(1-\eta_x)(s)ds$$
$$-\sum_{x=1}^{nN}(e^{-\lambda/nN}-1)\int_0^{N^2t}\eta_x(1-\eta_{x-1})(s)ds\Big\}$$

is a martingale with mean 1.

Then, the second and third terms in the exponent equal

$$\begin{aligned}
&\sum_{x=1}^{nN}\Big[(e^{\lambda/nN}-1)\int_0^{N^2t}\eta_{x-1}(1-\eta_x)(s)ds \\
&\qquad +(e^{-\lambda/nN}-1)\int_0^{N^2t}\eta_x(1-\eta_{x-1})(s)ds\Big] \\
&= \sum_{x=1}^{nN}\Big[(e^{\lambda/nN}-\lambda/nN-1)\int_0^{N^2t}\eta_{x-1}(1-\eta_x)(s)ds \\
&\qquad +(e^{-\lambda/nN}+\lambda/nN-1)\int_0^{N^2t}\eta_x(1-\eta_{x-1})(s)ds\Big] \\
&\qquad +\frac{\lambda}{nN^2}\int_0^{N^2t}(\eta_0-\eta_{nN})(s)ds \\
&\leq \frac{2e^{\lambda/nN}\lambda^2}{n^2N^2}(nN)(N^2t)+\frac{\lambda}{nN}(N^2t) \quad \leq \frac{C(t,\lambda)N}{n}
\end{aligned}$$

which gives the result with standard manipulations. □

3.3. **Exponential tightness estimate.** We now show that the scaled tagged particle positions are exponential tight.

Lemma 3.2. *Starting from local equilibrium measures or deterministic configurations, we have*

$$\lim_{a\uparrow\infty}\lim_{N\uparrow\infty}\frac{1}{N}\log P\Big\{|X_{N^2T}/N| \;\geq\; a\Big\} \;=\; -\infty.$$

Proof. From (3.3), we need only super-exponentially estimate, for a positive (as a similar argument works for $a<0$) and n fixed,

$$P\Big\{Y_T^N(G_n)-Y_0^N(G_n)+\frac{1}{nN^2}\sum_{x=1}^{nN}J_{x-1,x}(N^2T)\geq Y_T^N(1_{[0,a]})\Big\}.$$

We need only estimate

$$\begin{aligned}
&E\Big[\exp\Big\{N\Big[Y_T^N(G_n)-Y_0^N(G_n)+\frac{1}{nN^2}\sum_{x=1}^{nN}J_{x-1,x}(N^2T)-Y_T^N(1_{[0,a]})\Big]\Big\}\Big] \\
&\qquad = E[e^{Q_1}e^{Q_2}e^{Q_3}e^{Q_4}]
\end{aligned}$$

where $Q_1=NY_T^N(G_n)$, $Q_2=-NY_0^N(G_n)$, $Q_3=(nN)^{-1}\sum_{x=1}^{nN}J_{x-1,x}(N^2t)$, and $Q_4=-\sum_{x=0}^{\lfloor aN\rfloor}\eta_{N^2T}(x)$. By Chebychev, we can estimate the exponential terms separately. For fixed n, $\lim N^{-1}\log E[e^{4Q_3}]$ is bounded from Proposition 3.1, and as $Q_1 \leq nN$ by properties of G_n, $\lim N^{-1}\log E[e^{4Q_1}]$ is also bounded. In addition, as $\exp\{4Q_2\} \leq 1$, this term can be neglected.

Finally, by Borcea-Branden-Liggett [7][Theorem 5.2], as the initial measure of type (LEM) or (DIC) is a product measure (of degenerate Bernoulli's under (DIC) initial configurations), the coordinates $\{\eta_{N^2T}(x)\}$ are negatively associated. Hence, $E[e^{4Q_4}] \leq \prod_{x=1}^{\lfloor aN\rfloor} E[e^{-4\eta_{N^2T}(x)}]$, and using $\log(1-x) \leq -x$ for $0 \leq x \leq 1$, we write

$$\begin{aligned}\frac{1}{N}\log E[e^{4Q_4}] &\leq \frac{1}{N}\sum_{x=1}^{\lfloor aN\rfloor}\log E[e^{-4\eta_{N^2T}(x)}]\\ &\leq \frac{1}{N}\sum_{x=1}^{\lfloor aN\rfloor}\log\left[(e^{-4}-1)P(\eta_{N^2T}(x)=1)+1\right]\\ &\leq \frac{e^{-4}-1}{N}E\left[\sum_{x=1}^{\lfloor aN\rfloor}\eta_{N^2T}(x)\right] \ \to\ (e^{-4}-1)\int_0^a m(T,x)dx\end{aligned}$$

where $m(T,x) = \sigma_T * \gamma(x)$ is the solution of the hydrodynamic equation (Proposition 1.1). Since $\sigma_{T/10} * \gamma(x) \geq \gamma_* > 0$ as $\gamma \in M_1(\rho_*,\rho^*)$ for $\rho_*,\rho^* > 0$, the right-side is bounded above by $(e^{-4}-1)\gamma_* a \downarrow -\infty$ as $a \uparrow \infty$. □

3.4. **Weak LDP upperbounds.**

The weak upperbound, starting from local equilibrium measures or deterministic initial configuration, follows in several steps.

Step 1. Consider an interval $[a,b]$ for $0 < a < b$; subsequent arguments carry over straightforwardly to all intervals $[a,b] \subset \mathbb{R}$ using (3.2) by splitting at the origin if necessary. Now, divide $[a,b]$ into m equal intervals $A_k = [c_k, c_{k+1}]$. Then, by the union of events estimate,

$$\limsup_{N\to\infty}\frac{1}{N}\log P\Big(X_{N^2T}/N \in [a,b]\Big) \ \leq\ \max_k \limsup_{N\to\infty}\frac{1}{N}\log P\Big(X_{N^2T}/N \in A_k\Big).$$

Then, from (3.3) and Proposition 3.1, we have that

$$\begin{aligned}&\limsup_{N\uparrow\infty}\frac{1}{N}\log P\Big(X_{N^2T}/N \in [a,b]\Big)\\ &\quad\leq \limsup_{m\uparrow\infty}\limsup_{\delta\downarrow 0}\limsup_{n\uparrow\infty}\max_{1\leq k\leq m}\limsup_{N\uparrow\infty}\\ &\qquad \frac{1}{N}\log P\Big(Y_T^N(G_n) - Y_0^N(G_n) \in \Big[Y_T^N(1_{[0,c_k]}) - \delta, Y_T^N(1_{[0,c_{k+1}]}) + \delta\Big]\Big).\end{aligned}$$

Since the maps $\mu \mapsto \int G(x)\mu_T dx, \int G(x)\mu_0 dx, \int_0^c \mu_T dx$, for compactly supported G and constants c, are continuous in the Skorohod topology on $D([0,T];M_1)$, from Corollary 1.4, we conclude, for fixed k, n and δ that

$$\begin{aligned}&\limsup_{N\to\infty}\frac{1}{N}\log P\Big(Y_T^N(G_n) - Y_0^N(G_n)ds \in \Big[Y_T^N(1_{[0,c_k]}) - \delta, Y_T^N(1_{[0,c_{k+1}]}) + \delta\Big]\Big)\\ &\quad\leq -\inf\Big\{I_\gamma(\mu); \int G_n(x)\big[\mu_T(x) - \mu_0(x)\big]dx \qquad (3.4)\\ &\qquad\qquad \in \Big[\int_0^{c_k}\mu_T(x)dx - \delta, \int_0^{c_{k+1}}\mu_T(x)dx + \delta\Big]\Big\}.\end{aligned}$$

Step 2. Next, we give a uniform upperbound of the infimum in (3.4). We exhibit a density $\mu^{\mathfrak{c}}$ satisfying, for each $\delta > 0$ and all large n,

$$\int G_n(x)[\mu^{\mathfrak{c}}_T(x) - \mu^{\mathfrak{c}}_0(x)]dx \in \Big[\int_0^{\mathfrak{c}} \mu^{\mathfrak{c}}_T(x)dx - \delta, \int_0^{\mathfrak{c}} \mu^{\mathfrak{c}}_T(x)dx + \delta\Big]$$

and $\sup_{\mathfrak{c}\in[a,b]} I_\gamma(\mu^{\mathfrak{c}}) < B_0 < \infty$ where B_0 is independent of n and δ.

This is accomplished by the constructions in subsection 2.1, namely one can take $\mu^{\mathfrak{c}} = \sigma_t * \gamma + \lambda\epsilon(t/T)\psi(x/L)$ with λ and L chosen so that $\lambda L \int_{|\mathfrak{c}/L|}^1 \psi(x)dx = \int_{u_T}^{\mathfrak{c}} \sigma_T * \gamma(x)dx$. Let $J^{\mathfrak{c}}$ be its current, and $H^{\mathfrak{c}}_x$ be the associated function with respect to (1.3).

Proposition 2.1 gives $I_\gamma(\mu^{\mathfrak{c}})$ is uniformly bounded for $\mathfrak{c} \in [a,b]$. Now compute

$$\begin{aligned}
&\int G_n(x)\big[\mu^{\mathfrak{c}}_T(x) - \mu^{\mathfrak{c}}_0(x)\big]dx \qquad (3.5)\\
&= \int_0^T \int G_n(x)\left[(1/2)\partial_{xx}\mu^{\mathfrak{c}} - \partial_x H^{\mathfrak{c}}_x \mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})\right]dxdt\\
&= \int_0^T -(1/2)\partial_x \mu^{\mathfrak{c}}(t,0) + H^{\mathfrak{c}}_x\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})(t,0)dt\\
&\quad + \frac{1}{n}\int_0^T \int_0^n \big[(1/2)\partial_x\mu^{c} - H^{\mathfrak{c}}_x\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})\big]dxdt.
\end{aligned}$$

Since $\int_0^{\mathfrak{c}} \mu^{\mathfrak{c}}_T(x)dx = \int_0^T J^{\mathfrak{c}}(0,t)dt$ and $J^{\mathfrak{c}}(0,t) = -(1/2)\partial_x\mu^{\mathfrak{c}}(t,0) + H^{\mathfrak{c}}_x\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})(t,0)$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \sup_{\mathfrak{c}\in[a,b]} \Big|\int G_n(x)\big[\mu^{\mathfrak{c}}_T(x) - \mu^{\mathfrak{c}}_0(x)\big]dx - \int_0^{\mathfrak{c}} \mu^{\mathfrak{c}}_T(x)dx\Big|\\
&\leq \lim_{n\to\infty} \frac{1}{n}\Big|\int_0^T\int_0^n (1/2)\partial_x\mu^{\mathfrak{c}} - H^{\mathfrak{c}}_x\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})dxdt\Big|\\
&\leq \sup_{\mathfrak{c}\in[a,b]} \Big|\frac{1}{2n}\int_0^T (\mu^{\mathfrak{c}}_t(n) - \mu^{\mathfrak{c}}_t(0))dt\Big| + \frac{1}{\sqrt{n}}\|H^{\mathfrak{c}}_x\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})\|_{L^2([0,T]\times\mathbb{R})}.
\end{aligned}$$

Since $\|H^{\mathfrak{c}}_x\mu^{\mathfrak{c}}(1-\mu^{\mathfrak{c}})\|^2_{L^2} \leq 2I_0(\mu^{\mathfrak{c}})$, the right-side is $O(n^{-1/2})$ by Proposition 2.1.

Step 3. As I_γ is a good rate function, by the uniform bounds in step 2, out of minimizers $\nu^{k,n,\delta}$ over $k = k(m)$, and n, δ in the infimum in (3.4), by the uniform bound on $I_\gamma(\nu^{k,n,\delta})$, we can extract a subsequence, on which the limsup of (3.4) is attained as $\delta \downarrow 0$ and $n, m \uparrow \infty$, and which converges in $D([0,T];M_1)$ to a $\bar{\mu}$.

By Proposition 2.2, the subsequence, labeled $\nu^{k,n,\delta}$ itself for simplicity, may be approximated by $\{\mu^{k,n,\delta}\}$ so that $\mu^{k,n,\delta}$ is smooth, strictly bounded between 0 and 1, $H^{k,n,\delta}_x \in C^\infty([0,T]\times\mathbb{R})$, Skorohod distance $d(\mu^{k,n,\delta}, \nu^{k,n,\delta}) \downarrow 0$, $|I_0(\nu^{k,n,\delta}) - I_0(\mu^{k,n,\delta})| \downarrow 0$, and when $\hat{\gamma} \in M_1(\rho_*, \rho^*)$ is a.s. continuous and $0 < \hat{\gamma}(x) < 1$ for $x \in \mathbb{R}$, $|h(\nu^{k,n,\delta}_0; \hat{\gamma}) - h(\mu^{k,n,\delta}_0; \hat{\gamma})| \downarrow 0$. Also, as $[a,b]$ is compact, the subsequence can be chosen so that c_{k+1} converges to a $\mathfrak{c} \in [a,b]$.

Given $\nu^{k,n,\delta}$ satisfies the restriction in (3.4), we may also arrange

$$\begin{aligned}
\int_0^{c_k} \mu^{k,n,\delta}_T(x)dx - 2\delta &\leq \int G_n(x)\big[\mu^{k,n,\delta}_T(x) - \mu^{k,n,\delta}_0(x)\big]dx\\
&\leq \int_0^{c_{k+1}} \mu^{k,n,\delta}_T(x)dx + 2\delta. \qquad (3.6)
\end{aligned}$$

With these specifications, by lower semi-continuity, we have (3.4) is less than, in the case of starting from a local equilibrium measure,

$$\lim_{m\uparrow\infty}\lim_{\delta\downarrow 0}\lim_{n\uparrow\infty}\max_k -I^{LE}_\gamma(\mu^{k,n,\delta}) \ \leq\ -I^{LE}_\gamma(\bar\mu).$$

When starting from a deterministic configuration, noting $\nu_0^{k,n,\delta} = \bar\mu_0 = \gamma$, (3.4) is less than

$$\lim_{m\uparrow\infty}\lim_{\delta\downarrow 0}\lim_{n\uparrow\infty}\max_k -I_0(\mu^{k,n,\delta}) \ \leq\ -I_0(\bar\mu) \ =\ -I^{DC}_\gamma(\bar\mu).$$

Step 4. We now show that $\bar\mu$ satisfies

$$\int_0^T \bar J(0,t)dt \ =\ \int_0^{\mathfrak c}\bar\mu_T(x)dx. \tag{3.7}$$

As convergence in $D([0,T];M_1)$ implies $\mu^{k,n,\delta}_T \to \bar\mu_T$, $c_{k+1}-c_k = m^{-1}$, and $0\le \mu^{k,n,\delta}_T(x)\le 1$, we have both

$$\int_0^{c_k}\mu^{k,n,\delta}_T(x)dx, \int_0^{c_{k+1}}\mu^{k,n,\delta}_T(x)dx \to \int_0^{\mathfrak c}\bar\mu_T(x)dx.$$

Also, following the sequence (3.5),

$$\begin{aligned}&\int G_n(x)\big[\mu^{k,n,\delta}_T(x)-\mu^{k,n,\delta}_0(x)\big]dx \\ &\quad= \int_0^T J^{k,n,\delta}(0,t)dt \\ &\qquad + \frac{1}{n}\int_0^T\int_0^n \big[(1/2)\partial_x\mu^{k,n,\delta}_t - H^{k,n,\delta}_x\mu^{k,n,\delta}(1-\mu^{k,n,\delta})(t,x)\big]dxdt.\end{aligned} \tag{3.8}$$

Since $\|H^{k,n,\delta}_x\mu^{k,n,\delta}(1-\mu^{k,n,\delta})\|^2_{L^2} \le 2I_0(\mu^{k,n,\delta})$ is uniformly bounded, the last integral is bounded uniformly as $n^{-1}T + n^{-1/2}\sqrt{I_0(\mu^{k,n,\delta})}$. On the other hand, $\int_0^T J^{k,n,\delta}(0,t)dt \to \int_0^T \bar J(0,t)dt$ by Lemma 2.9.

Hence, noting (3.6), we obtain (3.7) immediately.

Step 5. Therefore,

$$\begin{aligned}&\limsup_{m\uparrow\infty}\limsup_{\delta\downarrow 0}\limsup_{n\uparrow\infty}\max_{1\le k\le m} \\ &\quad -\inf\Big\{I_\gamma(\mu); \int G_n(x)\big[\mu_T(x)-\mu_0(x)\big]dx \\ &\qquad\qquad \in \Big[\int_0^{c_k}\mu_T(x)dx-\delta, \int_0^{c_{k+1}}\mu_T(x)dx+\delta\Big]\Big\} \\ &\quad\le\ -I_\gamma(\bar\mu)\ \le\ -\min_{\mathfrak c\in[a,b]}\mathbb{I}(\mathfrak c).\end{aligned}$$

Step 6. The weak LDP upperbound, for compact $K\subset\mathbb{R}$,

$$\lim_{N\uparrow\infty}\frac{1}{N}P\Big(X_{N^2t}/N\in K\Big) \le -\inf_{a\in K}\mathbb{I}(a), \tag{3.9}$$

is now standard given $\mathbb{I}$ is lower semi-continuous (Lemma 2.11).

3.5. **LDP Lowerbound.** For the first step, the scheme for the weak upper bound is used. Let $O \subset \mathbb{R}$ be a nonempty open set, and suppose $a \in O$. We also assume $a > 0$ as a similar argument works for $a \leq 0$ by focusing on a subinterval to the left of the origin. Let $\epsilon > 0$ be such that $a - \epsilon > 0$ and $(a-\epsilon, a+\epsilon) \subset O$.

Then, for $\theta > 0$,

$$\begin{aligned}
&\lim_{N\uparrow\infty} \frac{1}{N}\log P\Big(X_{N^2T}/N \in O\Big) \\
&\geq \lim_{N\uparrow\infty}\frac{1}{N}P\Big(X_{N^2T}/N \in (a-\epsilon, a+\epsilon)\Big) \\
&\geq \lim_{n\uparrow\infty}\lim_{N\to\infty}\frac{1}{N}\log P\Big(Y^N_T(1_{[0,a-\epsilon]}) < Y^N_T(G_n) - Y^N_0(G_n) \qquad (3.10)\\
&\qquad + \frac{1}{nN^2}\sum_{x=1}^{nN} J_{x-1,x}(N^2T) < Y^N_T(1_{[0,a+\epsilon]}), Y^N_T(1_{[a-\epsilon,a+\epsilon]}) > \theta\Big).
\end{aligned}$$

From Proposition 3.1 and Corollary 1.4, (3.10) is greater than

$$\begin{aligned}
&\lim_{\theta\downarrow 0}\lim_{\delta\downarrow 0}\lim_{n\uparrow\infty}\lim_{N\to\infty}\frac{1}{N}\log P\Big(Y^N_T(1_{[0,a-\epsilon]}) + \delta \\
&\qquad < Y^N_T(G_n) - Y^N_0(G_n) < Y^N_T(1_{[0,a+\epsilon]}) - \delta, Y^N_T(1_{[a-\epsilon,a+\epsilon]}) > \theta\Big) \\
&\geq \lim_{\theta\downarrow 0}\lim_{\delta\downarrow 0}\lim_{n\uparrow\infty} -\inf\Big\{I_\gamma(\mu) : \int_0^{a-\epsilon}\mu_T(x)dx + \delta \qquad (3.11)\\
&\qquad < \int G_n(x)\big[\mu_T(x) - \mu_0(x)\big]dx < \int_0^{a+\epsilon}\mu_T(x)dx - \delta, \int_{a-\epsilon}^{a+\epsilon}\mu_T(x)dx > \theta\Big\}.
\end{aligned}$$

Now, for $\alpha > 0$, let $\bar\mu$ be a density such that $|I_\gamma(\bar\mu) - \mathbb{I}(a)| < \alpha$, and

$$\int_0^T \bar J(0,t)dt \ = \ \int_0^a \bar\mu_T(x)dx.$$

By the method used for (3.5) and (3.8) in the last section, through approximations of $\bar\mu$ with smooth μ^n by Proposition 2.2, we can show that

$$\lim_n \int_0^\infty G_n(x)\big[\bar\mu_T(x) - \bar\mu_0(x)\big]dx \ = \ \int_0^T \bar J(0,t)dt. \qquad (3.12)$$

We will need now to approximate $\bar\mu$ as follows to ensure a certain positivity. Let $\chi = \mu(s,x) = \sigma_s * \gamma + \lambda\epsilon(t/T)\psi(x/L)$ from subsection 2.1 where λ, L are chosen so that $\int_0^T J^\chi(0,t)dt = \int_0^a \chi_T(x)dx$. Recall $I_\gamma(\chi) < \infty$, and note (3.12), with χ and J^χ replacing $\bar\mu$ and J, also holds by the explicit construction. For $0 < b < 1$, define $\mu^b = (1-b)\chi + b\bar\mu$. Clearly, $\lim_{b\uparrow 1}\mu^b = \bar\mu$ uniformly, and so in $D([0,T]; M_1)$. In fact, $\lim_{b\uparrow 1} I_\gamma(\mu^b) = I_\gamma(\bar\mu)$: By lower semi-continuity, $\liminf I_\gamma(\mu^b) \geq I_\gamma(\bar\mu)$ and, by convexity, $\limsup I_\gamma(\mu^b) \leq I_\gamma(\bar\mu)$. Now, for given $\beta > 0$, let b be such that $|I_\gamma(\mu^b) - I_\gamma(\bar\mu)| < \beta$.

With $\theta > 0$, noting

$$\lim_n \int_0^\infty G_n(x)\big[\mu^b_T(x) - \mu^b_0(x)\big]dx \ = \ \int_0^a \mu^b_T(x)dx,$$

we have for $n \geq N(\theta, \bar{\mu}, \chi)$ that

$$\begin{aligned}
&\int_0^{a-\epsilon} \mu_T^b(x)dx + b\int_{a-\epsilon}^a \bar{\mu}_T(x)dx + (1-b)\int_{a-\epsilon}^a \chi_T(x)dx - \theta \\
&\quad\leq \int G_n(x)\big[\mu_T^b(x) - \mu_0^b(x)\big]dx \\
&\quad\leq \int_0^{a+\epsilon} \mu_T^b(x)dx - b\int_a^{a+\epsilon} \bar{\mu}_T(x)dx - (1-b)\int_a^{a+\epsilon} \chi_T(x)dx + \theta.
\end{aligned}$$

By the construction of χ, $\int_{a-\epsilon}^a \chi_T(x)dx, \int_a^{a+\epsilon} \chi_T(x)dx \geq c\epsilon$ for a constant $c > 0$. Hence, we can choose $\theta = \theta(\epsilon, b, \chi)$ so that for all small δ,

$$(1-b)\int_a^{a+\epsilon} \chi_T(x)dx - \theta,\ (1-b)\int_{a-\epsilon}^a \chi_T(x)dx - \theta \ > \ \delta.$$

Therefore, as $\bar{\mu}$ is nonnegative, μ^b satisfies the restriction in the infimum in (3.11). In particular, as $I_\gamma(\mu^b)$ does not depend on n, δ, θ, we have

$$\lim_{N\uparrow\infty} \frac{1}{N} P\Big(X_{N^2t}/N \in O\Big) \ \geq \ -I_\gamma(\mu^b) \ \geq \ -I_\gamma(\bar{\mu}) - \beta \ \geq \ -\mathbb{I}(a) - \alpha - \beta.$$

Hence,

$$\lim_{N\uparrow\infty} \frac{1}{N} P\Big(X_{N^2t}/N \in O\Big) \ \geq \ -\inf_{a\in O} \mathbb{I}(a). \tag{3.13}$$

3.6. **Proofs of Proposition 1.5 and Theorem 1.6.** First, we prove Theorem 1.6, and then prove Proposition 1.5.

Proof of Theorem 1.6. First, the function $\mathbb{I}$ is finite and lower semi-continuous on $\mathbb{R}$ by Proposition 2.1 and Lemma 2.11. Next, a 'weak' LDP is found from (3.9) and (3.13) with respect to rate function $\mathbb{I}$. Finally, standard arguments, given exponential tightness (Lemma 3.2), extend the 'weak' LDP to the full one in the theorem. □

Proof of Proposition 1.5. As noted in the proof of Theorem 1.6, $\mathbb{I}$ is a finite rate function on $\mathbb{R}$. Then, given the LDP in Theorem 1.6, and exponential tightness (Lemma 3.2), it follows $\mathbb{I}$ is a good rate function by Lemma 1.2.18 [9]. Divergence $\lim_{|a|\uparrow\infty} \mathbb{I}(a) = \infty$ is immediate.

When μ is such that $I_\gamma(\mu) < \infty$, the claims $\lim_L \int_0^L \mu_T(x) - \mu_0(x)dx$ converges and (1.6) are proved in Lemma 2.10.

Now, we note, in order for $I_\gamma(\mu) = 0$ we must have $h(\mu_0;\gamma) = 0$ and $I_0(\mu) = 0$. When $a = u_T$ as in (1.2), let μ be the solution of the heat equation with initial data γ. Since $\mu_0 = \gamma$, and as there is no asymmetry, $H_x^2\mu(1-\mu) = 0$. Hence, $h(\mu_0;\gamma) = I_0(\mu) = 0$, and we conclude $\mathbb{I}(u_T) = 0$.

On the other hand, when, $a \neq u_T$, if $\mathbb{I}(a)$ vanishes, out of a minimizing sequence of densities, through Propositions 2.2 and 2.9, one can find a convergent subsequence and extract a minimizing μ satisfying the restriction $\int_0^T J(0,t)dt = \int_0^a \mu_T(x)dx$, and by lower semi-continuity $h(\mu_0;\gamma) = I_0(\mu) = 0$. Then, $\mu_0 = \gamma$ a.s. and, noting (1.5), $H_x^2\mu(1-\mu) = 0$ a.s. In particular, $\mu_t = \sigma_t * \gamma$ is the unique bounded solution of the weak heat equation with initial data γ. However, as $\int_0^T J(0,t)dt = \int_0^{u_T} \mu_T(x)dx \neq \int_0^a \mu_T(x)dx$, but μ_T is positive and $a \neq u_T$, we obtain a contradiction. □

4. Asymptotic evaluations

We begin with some observations. Recall, when starting under a deterministic configuration with profile $\gamma(x) \equiv \rho$, in order for $I_\gamma^{\rm DC}(\mu) < \infty$, μ must satisfy $\mu_0(x) \equiv \rho$ and $I_\gamma^{\rm DC}(\mu) = I_0(\mu) < \infty$. Now, for such a density μ, which is also smooth, $\|\partial_x\|_{L^\infty} < \infty$, and strictly bounded away from 0 and 1, we have from Corollary 2.6 (applied with $\gamma(x) \equiv \rho$) that

$$I_0(\mu) \;=\; \frac{1}{8}\int\int_0^T \frac{(\partial_x\mu)^2}{\mu(1-\mu)}dtdx + \frac{1}{2}h(\mu_T;\rho) + \frac{1}{2}\int\int_0^T \frac{J^2}{\mu(1-\mu)}dtdx. \tag{4.1}$$

Also, when $\int_0^T J(0,t)dt = \int_0^a \mu_T(x)dx$, by the relation

$$\int_0^T J(0,t) - J(a,t)dt \;=\; \int_0^a \mu_T(x) - \mu_0(x)dx \;\text{ and }\; \int_0^T J(0,t)dt \;=\; \int_0^a \mu_T(x)dx,$$

we have

$$\int_0^T J(a,t)dt \;=\; \int_0^a \mu_0(x)dx \;=\; a\rho. \tag{4.2}$$

To calculate $\mathbb{I}(a)$ which is an infimum of $I_\rho^{DC}(\mu)$ over densities μ such that (4.2) holds, by translation-invariance, considering $\mu'(t,x) = \mu(t,x+a)$ and $J'(x,t) = J(x+a,t)$, we can replace (4.2) by the condition

$$\int_0^T J(0,t)dt \;=\; \int_a^{2a} \mu_0(x)dx \;=\; a\rho. \tag{4.3}$$

4.1. **Proof of Theorem 1.7.** We first prove the upperbound, and then the lowerbound.

Lemma 4.1. *For $a \in \mathbb{R}$, with respect to a constant c_1 depending on ρ,*

$$\mathbb{I}(a,\rho,T) \;\le\; c_1 \max\Big\{\frac{a^2}{\sqrt{T}}, \frac{|a|^3}{T}\Big\}.$$

Proof. We take $a > 0$ as the argument for $a < 0$ is symmetric. We recall the estimate (2.1) when $\mu_0 = \gamma = \rho$ and $\mathfrak{c} = a$:

$$\mathbb{I}(a,\rho,T) \;\le\; \frac{4\epsilon^*}{\rho(1-\rho)}\Big[\frac{\lambda^2 T}{4|L|}\int_{-1}^1 \psi'(x)^2dx + \frac{\lambda^2|L|^3}{T}\int_{-1}^1 \Psi(x)^2dx\Big] \tag{4.4}$$

where

$$\lambda \;=\; \frac{\rho a}{L\int_{|a/L|}^1 \psi(x)dx},$$

subject to the restriction $0 < \lambda \le \min\{\rho, 1-\rho\}/2$. The restriction will be satisfied as soon as $a/L \le 1/2$ and $a/L \le [\int_{1/2}^1 \psi(x)dx]\min\{1,(1-\rho)/\rho\}/2 := \kappa_0$, or $L \ge a\max\{2,\kappa_0^{-1}\}$.

Now take L in the form $L = \kappa\sqrt{T}$ for $\kappa \ge \kappa_1(a/\sqrt{T})$ where $\kappa_1 = \max\{2,\kappa_0^{-1}\}$. Substituting into (4.4), we obtain

$$\mathbb{I}(a,\rho,T) \;\le\; \frac{a^2}{\sqrt{T}}\frac{4\epsilon^*\rho}{1-\rho}\frac{\kappa}{\Big[\int_{1/2}^1 \psi(x)dx\Big]^2}\Big[\frac{1}{4\kappa^4}\int_{-1}^1 \psi'(x)^2dx + \int_{-1}^1 \Psi(x)^2dx\Big].$$

Hence, when $a/\sqrt{T}$ is large, say

$$\frac{a\kappa_1}{\sqrt{T}} \ \geq \ (4)^{-1/4}\Big[\int_{-1}^{1}(\psi')^2dx/\int_{-1}^{1}\Psi^2dx\Big]^{1/4} \ := \ \kappa_3,$$

we have

$$\mathbb{I}(a,\rho,T) \ \leq \ \frac{a^3}{T}\left[\frac{8\epsilon^*\rho\kappa_1}{1-\rho}\frac{\int_{-1}^1\Psi^2(x)dx}{\Big(\int_{1/2}^1\psi(x)dx\Big)^2}\right].$$

Correspondingly, when $a/\sqrt{T}$ is small, say $a\kappa_1/\sqrt{T}\leq\kappa_3$, we get

$$\mathbb{I}(a,\rho,T) \ \leq \ \frac{a^2}{\sqrt{T}}\left[\frac{8\epsilon^*\rho\kappa_3}{1-\rho}\frac{\int_{-1}^1\Psi^2(x)dx}{\Big(\int_{1/2}^1\psi(x)dx\Big)^2}\right].$$

It is not difficult now to fix the bounds so that, with respect to a constant c_1 depending on ρ, we obtain the desired bound on $\mathbb{I}(a,\rho,T)$. □

The lowerbound in Theorem 1.7 is implied by the following two estimates, relevant for large and small $|a|$.

Lemma 4.2. *For $a\in\mathbb{R}$,*

$$\mathbb{I}(a,\rho,T) \ \geq \ \frac{2|a|^3\rho^3}{3T}.$$

Lemma 4.3. *For $a\in\mathbb{R}$,*

$$\mathbb{I}(a,T,\rho) \ \geq \ \frac{a^2\rho^2}{\sqrt{6}\sqrt{T}}.$$

Proof of Lemma 4.2. Suppose $a>0$, as the argument for $a<0$ is similar. For $\epsilon>0$, by Proposition 2.2, let μ be a smooth density, bounded away from 0 and 1, such that $\|\partial_x\mu\|_{L^\infty}<\infty$, $\mu_0(x)\equiv\rho$, and

$$|I_0(\mu)-\mathbb{I}(a,\rho,T)| \ \leq \ \epsilon.$$

Noting Lemma 2.9 and (4.3), we can in addition impose

$$\Big|\int_0^T J(0,t)dt - a\rho\Big| \ \leq \ a\rho\epsilon.$$

Now, with the Lipschitz bound (cf. proof of Lemma 2.10),

$$\Big|\int_0^T J(x,t)dt-\int_0^T J(y,t)dt\Big| \ = \ \Big|\int_x^y\mu_T(z)-\mu_0(z)dz\Big| \ \leq \ |x-y|,$$

we have for $0\leq x\leq a\rho(1-\epsilon)$ that

$$\int_0^T J(x,t)dt \ \geq \ \int_0^T J(0,t)dt - x \ \geq \ a\rho(1-\epsilon)-x.$$

Write

$$\begin{aligned} a^3\rho^3(1-\epsilon)^3/3 \ &= \ \int_0^{a\rho(1-\epsilon)}[a\rho(1-\epsilon)-x]^2dx \\ &\leq \ \int_0^{a\rho(1-\epsilon)}\Big[\int_0^T J(x,t)dt\Big]^2dx \\ &\leq \ T\int\int_0^T J^2dtdx. \end{aligned}$$

Hence, as $\mu(1-\mu) \le 1/4$, from (4.1),

$$\mathbb{I}(a,\rho,T) \ \ge \ I_0(\mu) - \epsilon \ \ge \ 2\int\int_0^T J^2 dtdx - \epsilon \ \ge \frac{2a^3\rho^3(1-\epsilon)^3}{3T} - \epsilon.$$

As $\epsilon > 0$ is arbitrary, the lowerbound is recovered. □

Proof of Lemma 4.3. We start as in the proof of Lemma 4.2. However, we modify the initial step as follows:

$$\begin{aligned}\Big|\int_0^T J(x,t)dt - \int_0^T J(y,t)dt\Big| &= \Big|\int_x^y \mu_T(z) - \mu_0(z)dz\Big| \\ &\le 2\int_x^y |\sqrt{\mu_T(z)} - \sqrt{\mu_0(z)}|dz \\ &\le 2\sqrt{|x-y|}\Big(\int |\sqrt{\mu_T(z)} - \sqrt{\mu_0(z)}|^2 dz\Big)^{1/2} \\ &\le 2\sqrt{|x-y|}\sqrt{h(\mu_T;\rho)}\end{aligned}$$

where we use in the last line $\mu_0(x) \equiv \rho$ and the Hellinger inequality $h_d(\alpha;\beta) \ge (\sqrt{\alpha}-\sqrt{\beta})^2$. [Let $H(\alpha;\beta) = (\sqrt{\alpha}-\sqrt{\beta})^2 + (\sqrt{1-\alpha}-\sqrt{1-\beta})^2$. By Jensen inequality and $\log(1-x) \le x$ for $0 \le x < 1$, $h_d(\alpha;\beta) \ge -2\log[1-(1/2)H(\alpha;\beta)] \ge H(\alpha;\beta)$.]

Then, for $x \ge 0$, we have

$$\int_0^T J(x,t)dt \ \ge \ \int_0^T J(0,t)dt - 2\sqrt{xh(\mu_T;\rho)} \ \ge \ a\rho(1-\epsilon) - 2\sqrt{xh(\mu_T;\rho)}.$$

Let $\bar{h} = 2^2 h(\mu_T;\rho)$, and note $\bar{h} > 0$ as otherwise $\int_0^T \int J^2 dxdt = \infty$ which by Lemma 2.5 contradicts $I_\gamma(\mu) < \infty$. Then, for $0 \le x \le a\rho(1-\epsilon)$, we obtain

$$\begin{aligned}\frac{a^4\rho^4(1-\epsilon)^4}{6\bar{h}} &= \int_0^{a^2\rho^2(1-\epsilon)^2/\bar{h}} \big(a\rho(1-\epsilon) - \sqrt{x\bar{h}}\big)^2 dx \\ &\le \int_0^{a^2\rho^2(1-\epsilon)^2/\bar{h}} \Big(\int_0^T J(x,t)dt\Big)^2 dx \\ &\le T\int\int_0^T J^2(x,t)dtdx.\end{aligned}$$

As $\mu(1-\mu) \le 1/4$, from (4.1),

$$\mathbb{I}(a,\rho,T) \quad \ge \quad 2\int\int_0^T J^2(x,t)dtdx + \frac{1}{2}h(\mu_T;\rho) - \epsilon.$$

Then, with $A = \int\int_0^T J^2(x,t)dtdx$, we get

$$\mathbb{I}(a,\rho,T) \ \ge \ 2A + \frac{a^4\rho^4(1-\epsilon)^4}{48TA} - \epsilon.$$

Optimizing over A, and letting $\epsilon \downarrow 0$, we obtain $\mathbb{I}(a,\rho,T) \ge a^2\rho^2/(\sqrt{6}\sqrt{T})$ as desired. □

4.2. **Proof of Theorem 1.8.** We first make some useful reductions. By Proposition 2.2 and Lemma 2.9, for each $\epsilon > 0$, we can find a smooth density $\hat{\mu}$, strictly bounded away from 0 and 1, such that $\hat{\mu}_0(x) \equiv \rho$, $\|\partial_x\hat{\mu}\|_{L^\infty} < \infty$, and

$$\mathbb{I}(a,\rho,T) \ \ge \ I_0(\hat{\mu}) - \epsilon\frac{a^2}{\sqrt{T}} \quad \text{and} \quad \Big|\int_0^T \hat{J}(0,t)dt - a\rho\Big| < a\rho\epsilon.$$

Consider now a sequence $\{\mu^a\}$ of such $\epsilon a^2/\sqrt{T}$-minimizers of $\mathbb{I}(a,\rho,T)$ as $|a|\downarrow 0$. The upperbound in Lemma 4.1 gives $I_0(\mu^a)\to 0$, and Corollary 2.8 yields $\|\partial_x\mu^a\|_{L^2}\to 0$. Then, Lemma 2.4 shows that $\|\mu^a-\rho\|_{L^\infty}\to 0$. In particular, μ^a is strictly bounded away from 0 and 1 for all small a.

Then, we can approximate the entropy $h(\mu^a_T;\rho)$ as follows:

$$\Big|h(\mu^a_T;\rho)-\frac{1}{2\rho(1-\rho)}\int(\mu^a_T-\rho)^2dx\Big| \;\leq\; \epsilon(\|\mu^a-\rho\|_\infty)\int(\mu^a_T-\rho)^2dx$$

where $\epsilon(\alpha)\downarrow 0$ as $|\alpha|\downarrow 0$. By Hellinger's inequality $(\sqrt{\alpha}-\sqrt{\beta})^2\leq h_d(\alpha;\beta)$, we have also

$$\Big|h(\mu^a_T;\rho)-\frac{1}{2\rho(1-\rho)}\int(\mu^a_T-\rho)^2dx\Big| \;\leq\; 2\epsilon(\|\mu^a-\rho\|_\infty)h(\mu^a_T;\rho).$$

Similarly,

$$\Big|\frac{1}{\mu^a(1-\mu^a)}-\frac{1}{\rho(1-\rho)}\Big| \;\leq\; \frac{\epsilon(\|\mu^a-\rho\|)}{\mu^a(1-\mu^a)\rho(1-\rho)}$$

where again $\epsilon(\cdot)$ is a function where $\epsilon(\alpha)\downarrow 0$ as $|\alpha|\downarrow 0$.

Now, with respect to a smooth density μ, with $\mu_0(x)\equiv\rho$, define

$$K(t,x) \;=\; \int_0^t J(x,s)ds.$$

Then, as $\partial_t\mu+\partial_x J=0$,

$$\rho-\mu_t(x)=\partial_x K(t,x),\quad J(t,x)=\partial_t K(t,x),\quad\text{and}\quad \partial_x\mu(t,x)=\partial_{xx}K(t,x). \tag{4.5}$$

Denote

$$\mathcal{K} \;=\; \frac{1}{4}\int|K_x(T,x)|^2dx+\frac{1}{2}\int_0^T\int|K_t(t,x)|^2dxdt+\frac{1}{8}\int_0^T\int|K_{xx}(t,x)|^2dxdt.$$

Then, when $\|\mu-\rho\|_{L^\infty}$ is small, by (4.1),

$$\Big|I_0(\mu)-\frac{1}{\rho(1-\rho)}\mathcal{K}\Big| \;\leq\; \epsilon(\|\mu-\rho\|_{L^\infty})I_0(\mu).$$

Let's scale $cM(t,x)=K(tT,xL)$ with $L=\sqrt{T}$ and $c=\int_0^T J(t,0)dt$. Then, $\mathcal{K}$ becomes

$$\frac{c^2L}{4L^2}\int|M_x(1,x)|^2dx+\frac{c^2LT}{2T^2}\int_0^1\int|M_t(t,x)|^2dxdt$$
$$+\frac{c^2LT}{8L^4}\int_0^1\int|M_{xx}(t,x)|^2dxdt.$$

In other words,

$$\liminf_{|a|\downarrow 0}\frac{\sqrt{T}}{a^2}\mathbb{I}(a,\rho,T) \;\geq\; \frac{\rho}{(1-\rho)}\inf\mathcal{M}$$

where the infimum is over smooth M such that $M(0,x)\equiv 0$ and $M(1,0)=1$, and

$$\mathcal{M} \;=\; \frac{1}{4}\int|M_x(1,x)|^2dx+\frac{1}{2}\int_0^1\int|M_t(t,x)|^2dxdt+\frac{1}{8}\int_0^1\int|M_{xx}(t,x)|^2dxdt.$$

On the other hand, the upperbound

$$\limsup_{|a|\downarrow 0}\frac{\sqrt{T}}{a^2}\mathbb{I}(a,\rho,T) \;\leq\; \frac{\rho}{(1-\rho)}\inf\mathcal{M} \tag{4.6}$$

also follows by a similar strategy: In Proposition 4.4 below, we evaluate $\inf \mathcal{M}$ and find a minimizer. One can find a smooth ϵ approximating M with bounded derivatives and trace back through (4.5) to obtain the corresponding density μ^a with $\int_0^T J^a(0,t)dt = a\rho$ and $\mu_0^a(x) \equiv \rho$. Given $\|\partial_x M\|_{L^\infty([0,T]\times\mathbb{R})} < \infty$, we have $\|\mu^a - \rho\|_\infty \leq \|\partial_x K\|_{L^\infty} = (|a|\rho/\sqrt{T})\|\partial_x M\|_{L^\infty} = O(|a|)$. The argument to derive (4.6) now follows similar lines as for the lower bound.

Hence, the proof of Theorem 1.8 will follow from the evaluations $\inf_M \mathcal{M} = \sqrt{\pi}/2$, and $\sigma^2_{dyn} = (1-\rho)/(\rho\sqrt{\pi})$ in Propositions 4.4 and 4.5 below. ☐

Proposition 4.4. *We have*

$$\inf_M \mathcal{M} \;=\; \frac{\sqrt{\pi}}{2}.$$

Proof. Given convexity of $\mathcal{M}$, a minimizer exists. Now, we first minimize

$$\frac{1}{2}\int_0^1\int_{-\infty}^\infty |M_t(t,x)|^2 dxdt + \frac{1}{8}\int_0^1\int_{-\infty}^\infty |M_{xx}(t,x)|^2 dxdt$$

subject to $M(0,x)=0$ and $M(1,x)=f(x)$ where f is an arbitrary function. From the Euler equation, the minimizer solves

$$M_{tt} \;=\; \frac{1}{4}M_{xxxx}$$

with the boundary conditions at $t=0,1$.

One can verify the solution, in terms of Fourier transform with respect to the x variable but not transforming the t variable, is given by

$$\hat{M}(t,y) \;=\; \hat{M}(1,y)\frac{e^{\frac{ty^2}{2}} - e^{\frac{-ty^2}{2}}}{e^{\frac{y^2}{2}} - e^{\frac{-y^2}{2}}}$$

where

$$\hat{M}(1,y) \;=\; \frac{1}{\sqrt{2\pi}}\int e^{iyx}f(x)dx.$$

The minimum value, through Plancherel's formula, is then expressed as

$$\int_{-\infty}^\infty |\hat{M}(1,y)|^2 k(y)dy$$

where

$$\begin{aligned}
k(y) \;&=\; \int_0^1 \frac{y^4}{8}\frac{\left[e^{\frac{ty^2}{2}} + e^{\frac{-ty^2}{2}}\right]^2 + \left[e^{\frac{ty^2}{2}} - e^{\frac{-ty^2}{2}}\right]^2}{\left[e^{\frac{y^2}{2}} - e^{\frac{-y^2}{2}}\right]^2}dt \\
&=\; \int_0^1 \frac{y^4}{4}\frac{e^{ty^2} + e^{-ty^2}}{\left[e^{y^2/2} - e^{-y^2/2}\right]^2}dt \\
&=\; \frac{y^2}{4}\frac{e^{y^2} - e^{-y^2}}{\left[e^{y^2/2} - e^{-y^2/2}\right]^2} \;=\; \frac{y^2}{4}\frac{e^{y^2/2} + e^{-y^2/2}}{e^{y^2/2} - e^{-y^2/2}}.
\end{aligned}$$

Now, we add the term

$$\frac{1}{4}\int_{-\infty}^\infty |M_x(1,x)|^2 dx \;=\; \frac{1}{4}\int_{-\infty}^\infty y^2|\hat{M}(1,y)|^2 dy.$$

Together, the boundary minimization is to minimize

$$\frac{1}{4}\int_{-\infty}^{\infty} |\hat{M}(1,y)|^2 y^2\Big[1+\frac{e^{y^2/2}+e^{-y^2/2}}{e^{y^2/2}-e^{-y^2/2}}\Big]dy$$
$$= \frac{1}{2}\int_{-\infty}^{\infty} |\hat{M}(1,y)|^2 y^2 \frac{e^{y^2/2}}{e^{y^2/2}-e^{-y^2/2}}dy$$

subject to

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{M}(1,y)dy \;=\; 1.$$

We now recall the minimizer of

$$\int |g(y)|^2 K(y)dy$$

subject to $\int g(y)dy = a$ is given by $g(y) = cK(y)^{-1}$ with $c = a[\int K(y)^{-1}dy]^{-1}$, with minimum value $a^2[\int K(y)^{-1}dy]^{-1}$. Hence, with $a = \sqrt{2\pi}$ and

$$K(y) \;=\; \frac{y^2}{2}\frac{e^{y^2/2}}{e^{y^2/2}-e^{-y^2/2}}$$

so that

$$\int K(y)^{-1}dy \;=\; 2\int \frac{1-e^{-y^2}}{y^2}dy \;=\; 2\int_0^1\int e^{-ty^2}dydt \;=\; 2\int_0^1\sqrt{\frac{\pi}{t}}dt \;=\; 4\sqrt{\pi}.$$

Then, the infimum, $\inf_M \mathcal{M} = 2\pi/(4\sqrt{\pi}) = \sqrt{\pi}/2$ as desired. □

Proposition 4.5. *The dynamical portion of the limiting variance of $T^{-1/4}x(T)$ under ν_ρ is*

$$\sigma^2_{dyn} := \lim_{T\to\infty}\frac{1}{\sqrt{T}}E_{\nu_\rho}\Big[\Big(x(T)-E_\eta[x(T)]\Big)^2\Big] \;=\; \frac{1-\rho}{\rho}\frac{1}{\sqrt{\pi}}.$$

Proof. First, we recall part of the results in [1] that the limit distribution and variance of both $T^{-1/4}x(T)$ and $\rho^{-1}T^{-1/4}J_{-1,0}(T)$ are the same, namely $N(0,\sigma^2)$ with $\sigma^2 = \sqrt{2/\pi}(1-\rho)/\rho$. We now show

$$\lim_{T\to\infty}\frac{1}{\sqrt{T}}E_{\nu_\rho}\Big[\Big(E_\eta[J_{-1,0}(T)]\Big)^2\Big] \;=\; \rho(1-\rho)\frac{\sqrt{2}-1}{\sqrt{\pi}}.$$

Then, as

$$E_{\nu_\rho}[(J_{-1,0}(T))^2] \;=\; E_{\nu_\rho}[(J_{-1,0}(T)-E_\eta[J_{-1,0}(T)])^2]+E_{\nu_\rho}[(E_\eta[J_{-1,0}(T)])^2],$$

and the limiting variance of the current, $T^{-1/4}J_{-1,0}(T)$ under ν_ρ is $\rho(1-\rho)\sqrt{2}/\sqrt{\pi}$, we can obtain the desired result.

Now, the current $J_{-1,0}$ has martingale decomposition (cf. Section 2 [23]),

$$J_{-1,0}(t) \;=\; M(t)+\frac{1}{2}\int_0^t \eta_s(-1)-\eta_s(0)ds.$$

Also, for $x\in\mathbb{Z}$, from 'duality' (cf. Liggett [20][Section VIII.1 p. 363]),

$$E_\eta[\eta_t(x)] \;=\; \sum_i p(t,i-x)\eta(i)$$

where $p(t,j) = P(S_t = j)$ is the probability a continuous time random walk, starting from the origin, travels to j in time t. Then,

$$\begin{aligned} E_\eta[J_{-1,0}(T)] &= \frac{1}{2}\int_0^T E_\eta[\eta_t(-1)] - E_\eta[\eta_t(0)]dt \\ &= \frac{1}{2}\sum_i \eta(i)\int_0^T p(t,i+1) - p(t,i)dt \\ &= \frac{1}{2}\sum_i (\eta(i)-\rho)\int_0^T p(t,i+1) - p(t,i)dt. \end{aligned}$$

Therefore, from independence of coordinates $\{\eta(i)\}$,

$$Q_0(T) \ := \ E_{\nu_\rho}\Big[\Big(E_\eta[J_{-1,0}(T)]\Big)^2\Big] \ = \ \rho(1-\rho)\sum_i \Big|\frac{1}{2}\int_0^T p(t,i+1)-p(t,i)dt\Big|^2.$$

Now, as a priori the variance $Q_0(u) \leq E_{\nu_\rho}[J^2_{-1,0}(u)] = O(\sqrt{u})$, we need only find the limit of

$$\begin{aligned} Q_1(T) &= \frac{\rho(1-\rho)}{\sqrt{T}}\sum_i \Big|\frac{1}{2}\int_{\epsilon T}^T p(t,i+1)-p(t,i)dt\Big|^2 \qquad (4.7) \\ &= \frac{\rho(1-\rho)}{4\sqrt{T}}\sum_i \int_{[\epsilon T,T]^2} \big[p(t,i+1)-p(t,i)\big]\big[p(s,i+1)-p(s,i)\big]dsdt \end{aligned}$$

To estimate the integrand, from Doob's inequality, note

$$p(t,x) \ = \ \mathbb{E}[P(S_{N_t}=x)] \ = \ \mathbb{E}[P(S_{N_t}=x), \sup_{t\in[\epsilon T,T]}|N_t/t-1|\leq \epsilon] + O(T^{-10}) \quad (4.8)$$

where N_t is a Poisson process with rate 1 independent of the discrete time random walk $\{S_k\}$, and $\mathbb{E}$ refers to expectation with respect to N_t. Further, from the local limit theorem (Petrov [24][Theorem VII.13; p. 205]), uniformly over x, with respect to the discrete time walk, we have for $N_t \geq 1$ that

$$\begin{aligned} P(S_{N_t}=x) &= \frac{1}{\sqrt{2\pi N_t}}e^{-\frac{x^2}{2N_t}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2N_t}}\frac{q_2(x/\sqrt{N_t})}{N_t^{3/2}} + o(N_t^{-3/2}) \\ &= \frac{1}{\sqrt{2\pi N_t}}e^{-\frac{x^2}{2N_t}} + O(N_t^{-3/2}) \qquad (4.9) \end{aligned}$$

where $q_2(y) = (\gamma_4/24\theta^4)(y^4-6y^2+3)$, γ_k is the kth order cumulant and θ^2 is the variance of the symmetric Bernoulli variable. [In our case, in Petrov's formula, $q_1(y) = (\gamma_3/6\theta^3)(y^3-3y) \equiv 0$ as $\gamma_3 = 0$.]

Let $p^N(t,x) = P(S_{N_t}=x)$ and $p^R(s,x) = P(S_{R_s}=x)$ where R_s is an independent Poisson process also with rate 1. We now argue that only the leading terms in (4.8) and (4.9) are significant.

Since $\sum_x p(u,x), \sum_x p^N(u,x) \leq 1$, the error term on order $O(T^{-10})$ in (4.8) can be neglected in estimating (4.7). Indeed,

$$\frac{1}{\sqrt{T}}\int_{[\epsilon T,T]^2}\sum_i p(s,i)\cdot O(T^{-10})dsdt \ \leq \ \frac{T^2}{\sqrt{T}}\cdot O(T^{-10}) \ = \ o(1)$$

and

$$\frac{1}{\sqrt{T}}\int_{[\epsilon T,T]^2}\sum_i p^N(t,i)\cdot O(T^{-10})dsdt \ = \ o(1).$$

Also, note the error term of order $O(N_t^{-3/2})$ in (4.9) is not significant with respect to (4.7). Indeed,

$$\begin{aligned}
&\sum_x \frac{1}{\sqrt{2\pi N_t}}\left|e^{-\frac{(x+1)^2}{2N_t}} - e^{-\frac{x^2}{2N_t}}\right| \\
&\quad = \sum_x \frac{1}{\sqrt{2\pi N_t}}\left|e^{-(2x+1)/2N_t} - 1\right| e^{-x^2/2N_t} \\
&\quad \leq Ce^{-\sqrt{N_t}/4} + C\sum_{|x|\leq N_t^{3/4}} \frac{1}{\sqrt{2\pi N_t}}\frac{|x|}{N_t}e^{-x^2/2N_t} \ \leq\ \frac{C}{\sqrt{N_t}}
\end{aligned}$$

for some constants C. Then, given $|N_t/t-1|, |R_s/s-1| \leq \epsilon$ for $s,t\in[\epsilon T,T]$, a product of $\sum_i (2\pi N_t)^{-1/2}|e^{-(i+1)^2/2N_t} - e^{-i^2/2N_t}|$ and the error term with respect to the s-integration leads to bounding

$$\begin{aligned}
&\frac{1}{\sqrt{T}}\int_{[\epsilon T,T]^2}\sum_i \frac{1}{R_s^{3/2}}\frac{1}{\sqrt{2\pi N_t}}\left|e^{-\frac{(i+1)^2}{2N_t}} - e^{-\frac{i^2}{2N_t}}\right| \\
&\qquad\qquad \leq\ \frac{1}{\sqrt{T}}\int_{[\epsilon T,T]^2}\frac{C}{R_s^{3/2}\sqrt{N_t}}dsdt \ \leq\ O(T^{-1/2}).
\end{aligned}$$

Therefore, focusing on the leading order terms,

$$\begin{aligned}
&\frac{1}{4\sqrt{T}}\sum_i\int_{[\epsilon T,T]^2}\left[p^N(t,i+1)-p^N(t,i)\right]\left[p^R(s,i+1)-p^R(s,i)\right]dsdt \\
&\ =\ o(1) + \frac{1}{8\pi\sqrt{T}}\sum_i\int_{[\epsilon T,T]^2}\frac{1}{\sqrt{R_sN_t}}\left[e^{-(i+1)^2/2N_t} - e^{-i^2/2N_t}\right] \\
&\qquad\qquad\qquad\qquad \times\left[e^{-(i+1)^2/2R_s} - e^{-i^2/2R_s}\right]dsdt.
\end{aligned}$$

Now, using again $|N_t/t-1|, |R_s/s-1| \leq \epsilon$ for $s,t\in[\epsilon T,T]$, we further evaluate the integral on the right-side as

$$\begin{aligned}
&o(1) + \frac{1}{8\pi\sqrt{T}}\sum_{|i|\leq T^{3/4}}\int_{[\epsilon T,T]^2}\frac{i^2}{R_sN_t\sqrt{R_sN_t}}e^{\frac{-i^2}{2}\left[\frac{1}{N_t}+\frac{1}{R_s}\right]}dsdt \\
&\quad =\ o(1) + \frac{1}{8\pi\sqrt{T}}\int_{[\epsilon T,T]^2}\int_{-\infty}^{\infty}\frac{x^2}{R_sN_t\sqrt{R_sN_t}}e^{\frac{-x^2}{2}\left[\frac{1}{N_t}+\frac{1}{R_s}\right]}dxdsdt \\
&\quad =\ o(1) + \frac{\sqrt{2\pi}}{8\pi\sqrt{T}}\int_{[\epsilon T,T]^2}(R_sN_t)^{-3/2}\left(\frac{1}{N_t}+\frac{1}{R_s}\right)^{-3/2}dsdt \\
&\quad =\ o(1) + \frac{\sqrt{2}}{8\sqrt{\pi T}}\int_{[\epsilon T,T]^2}(N_t+R_s)^{-3/2}dsdt \ =:\ Q_2(T).
\end{aligned}$$

Finally, we have $Q_2(T)$ satisfies

$$\lim_{T\uparrow\infty}\left|Q_2(T) - \frac{\sqrt{2}-1}{\sqrt{\pi}}\right| \ \leq\ c(\epsilon)$$

where $c(\epsilon)$ vanishes as $\epsilon\downarrow 0$. □

4.3. **Proof of Theorem 1.9.** By symmetry,

$$P(|X(N^2T)|/N \geq a) \;=\; 2P(X(N^2T)/N \geq a)$$

for $a \geq 0$. For $0 \leq a \leq 1$, from (3.1) and noting $J_{-1,0}(t) - J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(t) = \sum_{x=0}^{aN}\eta_t(x) - \eta_0(x)$ by the development of subsection 3.1, we have

$$\left\{X(N^2t) \geq aN\right\} \;=\; \left\{J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(N^2t) \geq \sum_{x=0}^{\lfloor aN\rfloor}\eta_0(x)\right\}.$$

It will be helpful to rewrite currents in terms of a stirring process, details and constructions of which can be found in Chapter VIII [20]. Then,

$$J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(N^2t) \;=\; \sum_{x\leq\lfloor aN\rfloor}\eta_0(x)1_{[\xi^x_{N^2t}>\lfloor aN\rfloor]} - \sum_{x>\lfloor aN\rfloor}\eta_0(x)1_{[\xi^x_{N^2t}\leq\lfloor aN\rfloor]}.$$

Write, given the initial profile η_0 is deterministic, by Chebychev, that

$$\begin{aligned}\frac{1}{N}\log P\left(X(N^2t)\geq aN\right) \;&\leq\; \frac{1}{N}\log E\exp\left\{-\lambda\sum_{x=0}^{\lfloor aN\rfloor}\eta_0(x)\right\}\\ &\quad+\frac{1}{N}\log E\exp\left\{\lambda J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(N^2t)\right\}.\end{aligned}$$

The first term on the right-side tends to $-\lambda a$ as $N\uparrow\infty$. The second term is bounded, using Chebychev inequality, Liggett [20][Proposition VIII.1.7] noting $\exp\{\alpha\sum_{i=k}^{l}1_{[x_i\in A]}\}$ is positive definite for any $\alpha\in\mathbb{R}$, and $\log(1+x)\leq x$ for $|x|\leq 1$, as

$$\begin{aligned}&\frac{1}{N}\log E\exp\left\{\lambda J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(N^2t)\right\}\\ &\qquad\leq\; \frac{1}{2N}\log E\exp\left\{2\lambda\sum_{x\leq\lfloor aN\rfloor}\eta_0(x)1_{[\xi^x_{N^2t}>\lfloor aN\rfloor]}\right\}\\ &\qquad\quad+\frac{1}{2N}\log E\exp\left\{-2\lambda\sum_{x>\lfloor aN\rfloor}\eta_0(x)1_{[\xi^x_{N^2t}\leq\lfloor aN\rfloor]}\right\}\\ &\qquad\leq\; \frac{1}{2N}\sum_{x\leq\lfloor aN\rfloor}(e^{2\lambda\eta_0(x)}-1)P\left(\xi^x_{N^2t}>\lfloor aN\rfloor\right)\\ &\qquad\quad+\frac{1}{2N}\sum_{x>\lfloor aN\rfloor}(e^{-2\lambda\eta_0(x)}-1)P\left(\xi^x_{N^2t}\leq\lfloor aN\rfloor\right).\end{aligned}$$

Given $\eta_0(x)=1_{[|x|\leq N]}$ and $\xi^x_{N^2t}$ marginally is the position of a simple random walk started at x at time N^2t, as $N\uparrow\infty$, we have

$$\frac{1}{N}\sum_{x\leq\lfloor aN\rfloor}(e^{2\lambda\eta_0(x)}-1)P\left(\xi^x_{N^2t}>\lfloor aN\rfloor\right) \;\to\; \frac{e^{2\lambda}-1}{2}\int_{-1}^{a}P(N(0,t)>a-x)dx$$

and

$$\frac{1}{2N}\sum_{x>\lfloor aN\rfloor}(e^{-2\lambda\eta_0(x)}-1)P\left(\xi^x_{N^2t}\leq\lfloor aN\rfloor\right) \;\to\; \frac{e^{-2\lambda}-1}{2}\int_{a}^{1}P(N(0,t)\leq a-x)dx$$

where $N(0,t)$ is a Normal distribution with mean 0 and variance t.

Hence, combining the estimates, we have

$$\lim_{N\uparrow\infty}\frac{1}{N}\log P\left(X(N^2t)\geq aN\right) \leq -\lambda a+\frac{e^{2\lambda}-1}{2}\int_{-1}^{a}P(N(0,t)>a-x)dx + \frac{e^{-2\lambda}-1}{2}\int_{a}^{1}P(N(0,t)\leq a-x)dx.$$

Choosing $\lambda=\epsilon a$ for small $\epsilon>0$, we obtain

$$\lim_{N\uparrow\infty}\frac{1}{N}\log P\left(X(N^2t)\geq aN\right) \leq -\epsilon a^2\left[1-\frac{1}{a}\int_{1-a}^{1+a}P(N(0,t)>y)dy\right]+O(\epsilon^2) \leq -Ca^2$$

for a constant C noting $1>a^{-1}\int_{1-a}^{1+a}P(N(0,t)>y)dy$ for $0<a\leq 1$.

For $a\geq 1$, we write

$$J_{\lfloor aN\rfloor,\lfloor aN\rfloor+1}(t) = \sum_{|x|\leq N}\eta_0(x)1_{[\xi^x_{N^2t}>\lfloor aN\rfloor]}.$$

Then, as above,

$$\begin{aligned}P\left(X(N^2t)\geq aN\right) &\leq e^{-\lambda\sum_{x=0}^{\lfloor aN\rfloor}\eta_0(x)}E\exp\left\{\lambda\sum_{|x|\leq N}\eta_0(x)1_{[\xi^x_{N^2t}>\lfloor aN\rfloor]}\right\}\\ &\leq e^{-\lambda N}\prod_{|x|\leq N}E\exp\left\{\lambda 1_{[\xi^x_{N^2t}>\lfloor aN\rfloor]}\right\}.\end{aligned}$$

Taking logarithm, dividing by N, and taking the limit, we obtain

$$\begin{aligned}&\limsup_{N\uparrow\infty}\frac{1}{N}\log P\left(X(N^2t)\geq aN\right)\\ &\leq -\lambda+\limsup_{N\uparrow\infty}\frac{1}{N}\sum_{|x|\leq N}(e^{\lambda\eta_0(x)}-1)P(\xi^x_{N^2t}>\lfloor aN\rfloor)\\ &\leq -\lambda+(e^{\lambda}-1)\int_{-1}^{1}P(N(0,t)\geq a-x)dx.\end{aligned}$$

Optimizing on λ, we have the further bound

$$\begin{aligned}&\limsup_{N\uparrow\infty}\frac{1}{N}\log P\left(X(N^2t)\geq aN\right)\\ &\leq \log\int_{-1}^{1}P(N(0,t)>a-x)dx+1-\int_{-1}^{1}P(N(0,t)>a-x)dx < 0.\end{aligned}$$

However, for a large, the right-side is bounded by $-Ca^2/T$.

Working with the $0\leq a\leq 1$ and $a>1$ bounds, we obtain the desired quadratic order estimate. □

5. Proofs of Approximations

We give the proofs of Propositions 1.3 and 2.2, and Lemmas 2.3 and 2.4. Recall, $\sigma_t(x)=(2\pi t)^{-1/2}\exp\{-x^2/2t\}$.

5.1. **Proofs of Propositions 1.3 and 2.2.** To prove the approximations, we present a series of lemmas modeled on the approximation scheme in [17][p. 218-219] which refers to Propositions 3.4, 3.5 in Oelschläger [21]; see also Bertini-Landim-Mourragui [6] in this context. To this end, let μ be a density such that $I_0(\mu) < \infty$.

The first lemma states that finite rate densities μ are uniformly continuous in time, and is part of Lemma 3.4 [6] and Theorem 4.1 [16]. The proof is given here for the reader's convenience.

Lemma 5.1. *Let $\mathcal{J} \in C^2_K(\mathbb{R})$ be a function. Then, $s \mapsto \langle \mu_s, \mathcal{J}\rangle = \int \mathcal{J}(x)\mu_s(x)dx$ is a uniformly continuous function.*

Proof. Let $G \in C^{1,2}_K([0,T]\times\mathbb{R})$. As $I_0(\mu) < \infty$, from (1.4), we infer

$$l^2(\mu; G) \ \le\ 2I_0(\mu)\int_0^T \int G_x^2(t,x)\mu_t(x)(1-\mu_t(x))dxdt.$$

Let F be a smooth approximation of the indicator $1_{[s,t]}(u)$. Then, by applying the previous inequality with $G = F\mathcal{J}$, we obtain

$$\begin{aligned}&\left|\int \mu_t\mathcal{J}dx - \int \mu_s\mathcal{J}dx\right| \\ &\quad\le\ |t-s|\left\{\|\mathcal{J}''\|_{L^1} + \|\mathcal{J}'\|_{L^1}\right\} + 2I_0(\mu)|t-s|^{1/2}\|\mathcal{J}'\|_{L^2}.\end{aligned}$$

□

Lemma 5.2. *For each $\epsilon > 0$, there exists a density $\hat{\mu}$, smooth in space, such that the Skorohod distance $d(\hat{\mu};\mu) < \epsilon$, there is $0 < \delta_\epsilon < 1$ such that $\delta_\epsilon < \hat{\mu}(t,x) < 1-\delta_\epsilon$ for $(t,x) \in [0,T]\times\mathbb{R}$, and $|I_0(\hat{\mu}) - I_0(\mu)| < \epsilon$.*

In addition, if $\hat{\gamma} \in M_1(\rho_*,\rho^*)$ is a.s. continuous, $0 < \hat{\gamma}(x) < 1$ for $x \in \mathbb{R}$, and $h(\mu_0;\hat{\gamma}) < \infty$, then also $|h(\hat{\mu}_0;\hat{\gamma}) - h(\mu_0;\hat{\gamma})| < \epsilon$.

Proof. For $0 < \rho_*, \rho^* < 1$, let $\gamma \in M_1(\rho_*,\rho^*)$ be a function. Consider

$$\mu^{b,\alpha} \ =\ \sigma_{t+\alpha} * \gamma + b(\sigma_\alpha * \mu - \sigma_{t+\alpha} * \gamma) \tag{5.1}$$

for $0 \le b \le 1$ and $\alpha \ge 0$. Clearly, $\mu^{b,\alpha}$ is smooth in the space variable for $\alpha > 0$.

Next, for fixed $\alpha > 0$ and $0 < b < 1$, there is $0 < \delta < 1$ such that $\delta < \mu^{b,\alpha} < 1-\delta$ as $\sigma_{t+\alpha} * \gamma$ is strictly bounded between 0 and 1 for $t \in [0,T]$.

Now, $\mu^{b,\alpha} \to \mu^{1,\alpha}$ as $b \uparrow 1$ in $D([0,T]\times\mathbb{R})$. Also, noting $\lim_{\alpha\downarrow 0}\|\sigma_\alpha * G - G\|_{L^1(\mathbb{R})} = 0$ for $G \in L^1(\mathbb{R})$, we have $\mu^{1,\alpha} \to \mu$ as $\alpha \downarrow 0$.

By lower semi-continuity of I_0,

$$\liminf I_0(\mu^{b,\alpha}) \ \ge\ I_0(\mu).$$

On the other hand, by convexity of $I_0(\nu)$, we have

$$I_0(\mu^{b,\alpha}) \quad\le\quad (1-b)I_0(\sigma_{t+\alpha} * \gamma) + bI_0(\sigma_\alpha * \mu).$$

Note that $I_0(\sigma_{t+\alpha} * \gamma) = 0$, and by translation-invariance $I_0(\nu(t,x-y)) = I_0(\nu(t,x))$. Then, by convexity again, the right-side is less than

$$b\int \sigma_\alpha(y)I_0(\mu(t,x-y))dy \ =\ bI_0(\mu) \ \uparrow\ I_0(\mu).$$

Similarly, if $\hat{\gamma} \in M_1(\rho_*,\rho^*)$ is a.s. continuous and $0 < \hat{\gamma} < 1$, and $h(\mu_0;\hat{\gamma}) < \infty$, by lower semi-continuity and convexity of $h(\nu_0;\hat{\gamma})$ we have

$$h(\mu_0;\hat{\gamma}) \le \liminf h(\mu_0^{b,\alpha};\hat{\gamma}) \ \text{ and }\ h(\mu_0^{b,\alpha};\hat{\gamma}) \le (1-b)h(\sigma_\alpha * \gamma;\hat{\gamma}) + bh(\sigma_\alpha * \mu_0;\hat{\gamma}).$$

Also, once more by convexity,

$$h(\sigma_\alpha * \mu_0;\hat{\gamma}) \ \leq \ \int dy\sigma_\alpha(y)\int dx h_d(\mu_0(x);\hat{\gamma}(x-y)).$$

The right-side, since $h(\mu_0(\cdot);\hat{\gamma}(\cdot - y)) \leq C(1+|y|)$ by properties of $\hat{\gamma}$, converges to $h(\mu_0;\hat{\gamma})$. By the same argument, $h(\sigma_\alpha * \gamma;\hat{\gamma}) \to h(\gamma;\hat{\gamma})$. Hence, $\lim h(\mu^{b,\alpha};\hat{\gamma}) = h(\mu;\hat{\gamma})$.

Then, we may take $\hat{\mu} = \mu^{b,\alpha}$ for b close to 1 and α near 0. □

Lemma 5.3. *Let $\hat{\mu}$ be the density constructed in Lemma 5.2. For each $\epsilon > 0$, there exists a smooth density $\tilde{\mu}$ such that $\tilde{\mu}_0 = \hat{\mu}_0$, the Skorohod distance $d(\tilde{\mu};\hat{\mu}) < \epsilon$, and $|I_0(\tilde{\mu}) - I_0(\hat{\mu})| < \epsilon$. Also, all partial derivatives of $\tilde{\mu}$ are uniformly bounded in $[0,T]\times\mathbb{R}$.*

Proof. We first approximate $\hat{\mu}$ by smoothing in the time variable. Define for $\beta > 0$ a density which is constant in time on a short time interval:

$$\nu^\beta(t,x) \ = \ \begin{cases} \hat{\mu}_0(x) & \text{for } 0 \leq t < \beta \\ \hat{\mu}(t-\beta,x) & \text{for } \beta \leq t \leq T+\beta. \end{cases}$$

Let $\kappa_\varepsilon \in C_K^\infty(\mathbb{R})$ be smooth approximations of the identity in $L^1(\mathbb{R})$ such that $\kappa_\varepsilon \geq 0$, $\int \kappa_\varepsilon(x)dx = 1$, $\mathrm{Supp}(\kappa_\varepsilon) \subset (0,\varepsilon)$ and for $f \in L^1(\mathbb{R})$, $f * \kappa_\varepsilon \to f$ as $\varepsilon \downarrow 0$ in L^1. Form the convolution, for $0 < \varepsilon \leq \beta$,

$$\nu^{\beta,\varepsilon}(t,x) \ = \ \int_0^T \nu^\beta(t+s,x)\kappa_\varepsilon(s)ds.$$

It is clear, by continuity of $\hat{\mu}$ in time (Lemma 5.1), that $\lim_{\beta\downarrow 0}\lim_{\varepsilon\downarrow 0}\nu^{\beta,\varepsilon} = \hat{\mu}$ in $D([0,T];M_1)$. By construction, $\nu^{\beta,\varepsilon}$ is smooth.

Also, $\nu_0^{\beta,\varepsilon} = \hat{\mu}_0$.

From lower semi-continuity and convexity

$$\liminf_{\beta,\varepsilon} I_0(\nu^{\beta,\varepsilon}) \geq I_0(\hat{\mu}), \ \text{ and } \ I_0(\nu^{\beta,\varepsilon}) \leq \int_0^T \kappa_\varepsilon(s) I_0(\nu^\beta(t+s,x))ds.$$

Using the variational definition of I_0, noting $\hat{\mu}_0 = \sigma_\alpha * (\gamma + b(\mu_0 - \gamma))$, the rate of ν^β on the interval $[0,\beta]$ is bounded by

$$\begin{aligned} \sup_{G\in C_K^{1,2}} \frac{1}{2}\int_0^\beta \int G_x\partial_x\hat{\mu}_0 - G_x^2\hat{\mu}_0(1-\hat{\mu}_0)dxdt \ &\leq \ \frac{\beta}{8}\int \frac{(\partial_x\hat{\mu}_0)^2}{\hat{\mu}_0(1-\hat{\mu}_0)}dx \\ &\leq \ C(b,\alpha,\gamma)\beta \end{aligned}$$

which vanishes as $\beta \downarrow 0$ (cf. proof of Lemma 4.3 [6]). On the other hand, by the formula (1.5), the rate of ν^β on the interval $[\beta, T]$ converges to $I_0(\hat{\mu})$ as $\beta \downarrow 0$. We can conclude that $\lim_{\beta,\varepsilon\downarrow 0} I_0(\nu^{\beta,\varepsilon}) \to I_0(\hat{\mu})$.

With β,ε small, we can take $\tilde{\mu} = \nu^{\beta,\varepsilon}$.

Finally, since $\tilde{\mu}$ is smoothed in the space variable by a convolution with σ_α, smoothed in time by convolution with κ_ε, $\|\mu\|_{L^\infty} \leq 1$, and $\|\sigma_\alpha^{(k)}\|_{L^1(\mathbb{R})}, \|\kappa_\varepsilon^{(l)}\|_{L^1(\mathbb{R})} < \infty$, we have $\|\partial_x^{(k)}\partial_t^{(l)}\tilde{\mu}\|_{L^\infty} \leq \|\sigma_\alpha^{(k)}\|_{L^1}\|\kappa_\varepsilon^{(l)}\|_{L^1} < \infty$. □

Lemma 5.4. *Let $\tilde{\mu}$ be the density constructed in Lemma 5.3, and let $\tilde{H}_x$ be the function associated via (1.3). Then, $\tilde{H}_x \in C^\infty([0,T]\times\mathbb{R})$, $\|\tilde{H}_x\|_{L^2([0,T]\times\mathbb{R})} < \infty$, and $\|\tilde{H}_x\|_{L^\infty([0,T]\times\mathbb{R})} < \infty$.*

Proof. By construction, for a $\delta > 0$, the density $\tilde{\mu}$ satistifes $\delta < \tilde{\mu} < 1-\delta$, and is smooth with uniformly bounded derivatives on $[0,T]\times\mathbb{R}$ of all orders, and

$$\partial_t \tilde{\mu} \;=\; \frac{1}{2}\partial_{xx}\tilde{\mu} - \partial_x\big[\tilde{H}_x \tilde{\mu}(1-\tilde{\mu})\big]. \tag{5.2}$$

Then, as $2I_0(\tilde{\mu}) = \int_0^T \int (\tilde{H}_x)^2\tilde{\mu}(1-\tilde{\mu})dxdt < \infty$, we obtain the L^2 bound on $\tilde{H}_x$. Also, by solving for $\tilde{H}_x$ in (5.2), we have $\tilde{H}_x$ is smooth.

To deduce that $\tilde{H}_x$ is bounded, we note from (5.2) that $\tilde{H}_x\hat{\mu}(1-\hat{\mu})$ is Lipschtiz in the space variable x with uniform constant over $[0,T]\times\mathbb{R}$ as $\partial_t\tilde{\mu}$ and $\partial_{xx}\tilde{\mu}$ are bounded. The desired L^∞ bound now follows from the L^2 bound and $\delta < \hat{\mu} < 1-\delta$ once we show that $\tilde{H}_x\tilde{\mu}(1-\tilde{\mu})$ is also Lipschitz in the time variable t uniformly over $[0,T]\times\mathbb{R}$. To this end, write

$$\tilde{H}_x\tilde{\mu}(1-\tilde{\mu})(x,t) = \tilde{H}_x\tilde{\mu}(1-\tilde{\mu})(0,t) + (1/2)\partial_x\tilde{\mu}(x,t) - (1/2)\partial_x\tilde{\mu}(0,t) - \int_0^x \partial_t\tilde{\mu}dy.$$

The first three terms on the right-side are clearly uniformly Lipschitz in $t\in[0,T]$.

For the last term, consider a smooth G compactly supported in $[-\epsilon, x+\epsilon]$ which equals 1 on $[0,x]$. We have, as $\partial_{tt}\tilde{\mu}$ is bounded,

$$\Big|\int_0^x \partial_{tt}\tilde{\mu}(u,y)dy\Big| \;\le\; \Big|\int G(y)\partial_{tt}\tilde{\mu}(u,y)dy\Big| + 2C\epsilon.$$

Now, by construction in the proof of Lemma 5.3,

$$\int G(y)\partial_{tt}\tilde{\mu}(u,y)dy \;=\; \int_0^T \int G(y)\kappa_\varepsilon''(s)\nu^\beta(u+s,y)dyds.$$

As $I_0(\nu^\beta) < \infty$, we can associate via (1.3) an H_x^β to the density ν^β. From the weak formulation (1.4), and $\kappa_\varepsilon'(0) = \kappa_\varepsilon'(T) = 0$, the right-side equals

$$-\frac{1}{2}\int_0^T \int G''(y)\kappa_\varepsilon'(s)\nu^\beta(u+s,y)dyds$$
$$- \int_0^T \int G'(y)\kappa_\varepsilon'(s)H_x^\beta\nu^\beta(1-\nu^\beta)(u+s,y)dyds.$$

The first integral, because ν^β is bounded and $G' \neq 0$ on a set of width at most 2ϵ, and the second, as also $\|H_x^\beta\nu^\beta(1-\nu^\beta)\|_{L^2} \le 2I_0(\nu^\beta) < \infty$, are both uniformly bounded in u and x. □

The function $\tilde{H}_x$ associated to $\tilde{\mu}$ in Lemma 5.4, although smooth, does not necessarily have compact support. Let $H_x^m \in C_K^\infty((0,T]\times\mathbb{R})$ be smooth approximations of $\tilde{H}_x$ with the following properties: $\|H_x^m - \tilde{H}_x\|_{L^2([0,T]\times\mathbb{R})} \le m^{-1}$, and $\sup_{\substack{t\in[0,T]\\ x\in[-m,m]}} |H_x^m - \tilde{H}_x| \le m^{-1}$.

Let $w^m \in D([0,T];M_1)$ be the smooth density with initial condition $w_0^m = \tilde{\mu}_0$ and satisfying the equation

$$\partial_t v \;=\; (1/2)\partial_{xx}v - \partial_x(H_x^m v(1-v)).$$

Existence, for instance, follows from the hydrodynamic limit for weakly asymmetric exclusion processes in [16] using the replacement estimates Theorem 6.1 and Claims 1,2 [19][Section 6]; see also Theorem 3.1 [17]. Uniqueness in the class of bounded solutions follows by the method of Proposition 3.5 [21].

Lemma 5.5. *The sequence w^m converges uniformly to $\tilde{\mu}$ on compact subsets of $[0,T]\times\mathbb{R}$, and hence in $D([0,T];M_1)$. Also, $I_0(w^m)\to I_0(\hat{\mu})$.*

Proof. Suppose that we have proven $w^m\to\tilde{\mu}$ uniformly on compact subsets. As $\|H^m_x-\hat{H}_x\|_{L^2}\to 0$, we would then conclude $I_0(w^m)\to I_0(\tilde{\mu})$.

In the following, the constant C may change line to line. Now, given $\partial_t\sigma_t(x) = (1/2)\partial_{xx}\sigma_t(x)$, we have, for $t,h>0$, that

$$\begin{aligned}&\sigma_h * w^m_t(y) - \sigma_{t+h}*w^m_0(y)\\ &\quad = \int_0^t\int H^m_x w^m(1-w^m)(s,z)\frac{-(z-y)}{t+h-s}\sigma_{t-s+h}(z-y)dzds.\end{aligned}$$

By properties of w^m, H^m_x and $(|z-y|/\sqrt{u})\exp(-(z-y)^2/4u)\le 1$,

$$\begin{aligned}&|H^m_x w^m(1-w^m)|(s,z)\frac{|z-y|}{t+h-s}\sigma_{t+h-s}(z-y)\\ &\qquad\le\ C|t-s|^{-1/2}\sigma_{2T}(z-y)\ \in\ L^1([0,t]\times\mathbb{R}).\end{aligned}$$

Hence, taking $h\downarrow 0$, we obtain

$$w^m_t(y)\ =\ \sigma_t * w^m_0(y) + \int_0^t\int H^m_x w^m(1-w^m)(s,z)\frac{-(z-y)}{t-s}\sigma_{t-s}(z-y)dzds. \quad (5.3)$$

The equation (5.3) also holds with respect to $\tilde{\mu}$.

We have then, using again $(|z-y|/\sqrt{u})\exp(-(z-y)^2/4u)\le 1$, and $w^m_0=\tilde{\mu}_0$, that

$$\begin{aligned}&|w^m_t(y)-\tilde{\mu}_t(y)|\\ &\le\ \sigma_t * |w^m_0-\tilde{\mu}_0|(y) \qquad (5.4)\\ &+\int_0^t\int \big|H^m_x w^m(1-w^m)-\hat{H}_x\tilde{\mu}(1-\tilde{\mu})\big|(s,z)\frac{|z-y|}{t-s}\sigma_{t-s}(z-y)dzds\\ &\le C\int_0^t\int\big|H^m_x w^m(1-w^m)-\tilde{H}_x\tilde{\mu}(1-\tilde{\mu})\big|(s,z)(t-s)^{-1/2}\sigma_{2(t-s)}(z-y)dzds.\end{aligned}$$

We now estimate the last line in two parts, noting

$$\begin{aligned}&\big|H^m_x w^m(1-w^m)-\tilde{H}_x\tilde{\mu}(1-\tilde{\mu})\big|(s,z)\\ &\quad\le\ |H^m_x-\tilde{H}_x|(s,z)+|\tilde{H}_x|\big|\tilde{\mu}(1-\hat{\mu})-w^m(1-w^m)\big|(s,z).\end{aligned}$$

The first part, noting $|y|\le m/2$, by properties of H^m_x, $\|\sigma_t(x)\|_{L^2([0,T]\times\mathbb{R})}\le CT^{1/4}$, and $\sup_{t\in(0,T]}t^{-1/2}\sigma_{4t}(1)\le C$, is bounded for large m,

$$\begin{aligned}&\int_0^t\int|H^m_x-\tilde{H}_x|(s,z)(t-s)^{-1/2}\sigma_{2(t-s)}(z-y)dzds\\ &\le\ \int_0^t\int_{|z|\ge m}|H^m_x-\tilde{H}_x|(s,z)(t-s)^{-1/2}\sigma_{2(t-s)}(z-y)dzds+m^{-1}\sqrt{t}\\ &\le\ C\int_0^t\int_{|z|\ge m}|H^m_x-\tilde{H}_x|(s,z)\sigma_{4(t-s)}(z-y)dzds+m^{-1}\sqrt{t}\\ &\le\ Cm^{-1}T^{1/4}+m^{-1}\sqrt{T}.\end{aligned}$$

The second part,

$$\begin{aligned}\int_0^t \int |\tilde{H}_x|\big|\tilde{\mu}(1-\tilde{\mu}) - w^m(1-w^m)\big|(s,z)(t-s)^{-1/2}\sigma_{2(t-s)}(z-y)dzds \\ = \ D_1 + D_2 + D_3\end{aligned}$$

where D_1, D_2, D_3 is the integral over $[0,t]\times\{|z|\geq m/2+\epsilon\}$, $[0,t]\times\{m/2\leq|z|\leq m/2+\epsilon\}$ and $[0,t]\times\{|z|\leq m/2\}$ respectively, for $\epsilon>0$.

The term D_1, noting $\sup_{t\in(0,T]} t^{-1/2}\sigma_{4t}(\epsilon)\leq C_\epsilon$, is bounded by

$$\begin{aligned}2\int_0^t \int_{|z|\geq m/2+\epsilon} |\tilde{H}_x|(s,z)(t-s)^{-1/2}\sigma_{2(t-s)}(z-y)dzds \\ \leq \ C(\epsilon,T)\|\tilde{H}_x\|_{L^2([0,T]\times\{z:|z|\geq m/2\})}.\end{aligned}$$

The second term D_2 is bounded by

$$\begin{aligned}2\int_0^t \int_{m/2\leq|z|\leq m/2+\epsilon} |\tilde{H}_x||t-s|^{-1/2}\sigma_{2(t-s)}(z-y)dzds \\ \leq \ C\|\tilde{H}_x\|_{L^\infty}\int_0^t t^{3/4}\int_0^\epsilon t^{-1/4}e^{-z^2/4t}dzds \\ \leq \ C\|\tilde{H}_x\|_{L^\infty}T^{1/4}\sqrt{\epsilon}.\end{aligned}$$

The third term D_3 is bounded

$$\begin{aligned}2\int_0^t \int_{|z|\leq m/2} |\tilde{H}_x||\tilde{\mu}-w^m|(s,z)|t-s|^{-1/2}\sigma_{2(t-s)}(z-y)dzds \\ \leq \ 2\sqrt{t}\|\tilde{H}_x\|_{L^\infty}\sup_{\substack{|z|\leq m/2\\ s\leq t}}|\tilde{\mu}_s(z)-w^m_s(z)|.\end{aligned}$$

Hence, for $\tau>0$ small enough but fixed,

$$\begin{aligned}&\sup_{\substack{|z|\leq L\\ s\leq\tau}}|\tilde{\mu}(z,s)-w^m(z,s)| \\ &\leq \sup_{\substack{|z|\leq m/2\\ s\leq\tau}}|\tilde{\mu}(z,s)-w^m(z,s)| \\ &\leq \ C(T)m^{-1} + 2C(\epsilon,T)\|\tilde{H}_x\|_{L^2([0,T]\times\{z:|z|\geq m/2\})} + 2C\|\tilde{H}_x\|_{L^\infty}T^{1/4}\sqrt{\epsilon}.\end{aligned}$$

We may repeat the same scheme, starting from time τ, where now the initial difference (5.4) is taken into account:

$$\begin{aligned}&\sup_{|y|\leq m/3}\sigma_t * |w^m_\tau - \tilde{\mu}_\tau|(y) \\ &\leq \sup_{|z|\leq m/2}|w^m_\tau-\tilde{\mu}_\tau|(z) + \sup_{|y|\leq m/3}\int_{|z|>m/2}\sigma_t(y-z)dz \\ &\leq \sup_{|z|\leq m/2}|w^m_\tau-\tilde{\mu}_\tau|(z) + e^{-Cm^2/T}.\end{aligned}$$

With a finite number of iterations of such type, we obtain uniform convergence for $|z|\leq L$ and $0\leq s\leq T$. □

Lemma 5.6. *Recall w^m from Lemma 5.5. Suppose $\mu_0=\gamma\in M_1(\rho_*,\rho^*)$ for $0<\rho_*,\rho^*<1$. Then, for $\epsilon>0$, there exists M such that for $m\geq M$, there is a* *density $\bar{\chi} \in C^\infty((0,T]\times\mathbb{R})$ satisfying (1.3) with respect to $\bar{H}_x \in C^\infty_K([0,T]\times\mathbb{R})$ such that $\bar{\chi}_0 = \gamma$, the Skorohod distance $d(\bar{\chi}, w^m) < \epsilon$, and $|I_0(\bar{\chi}) - I_0(w^m)| < \epsilon$.*

Proof. Consider w^m_0 from Lemma 5.5. From the assumption $\mu_0 = \gamma$, we have $w^m_0 = \tilde{\mu}_0 = \sigma_\alpha * \gamma$ from (5.1). Form the density $\bar{\chi}$ as follows.

$$\bar{\chi} \;=\; \begin{cases} \sigma_t * \gamma & \text{for } 0 \le t \le \alpha \\ w^m_{t-\alpha} & \text{for } \alpha \le t \le T. \end{cases}$$

By construction $\bar{\chi} \in C^\infty((0,T]\times\mathbb{R})$. Also, noting H^m_x is supported on a compact subset of $(0,T]\times\mathbb{R}$, $\bar{\chi}$ satisfies (1.3) with respect to $\bar{H}_x \in C^\infty_K([0,T]\times\mathbb{R})$ given by

$$\bar{H}_x \;=\; \begin{cases} 0 & \text{for } (t,x) \in [0,\alpha]\times\mathbb{R} \\ H^m_x(t-\alpha, x) & \text{for } (t,x)\in[\alpha,T]\times\mathbb{R}. \end{cases}$$

Now,

$$2I_0(\bar{\chi}) \;=\; \int_0^T \int \bar{H}_x^2 \bar{\chi}(1-\bar{\chi})dxdt \;=\; \int_0^{T-\alpha}\int (H^m_x)^2 w^m(1-w^m)dxdt.$$

Then, the difference

$$2I_0(\bar{\chi}) - 2I_0(w^m) \;=\; \int_{T-\alpha}^T \int (H^m_x)^2 w^m(1-w^m)dxdt.$$

To estimate the right-side, recall from Lemma 5.5 that $\|H^m_x - \tilde{H}_x\|_{L^2} \le m^{-1}$, and $w^m \to \tilde{\mu}$ uniformly on compact subsets. Then,

$$\begin{aligned}
&\int_{T-\alpha}^T \int (H^m_x)^2 w^m(1-w^m)dxdt \\
&\le\; 2\|H^m_x - \tilde{H}_x\|^2_{L^2([0,T]\times\mathbb{R})} + 2\int_0^T \int_{|x|\ge L} \tilde{H}_x^2 dxdt \\
&\qquad +4\int_0^T\int_{|x|\le L} \tilde{H}^2_x |w^m - \tilde{\mu}|dxdt + 2\int_{T-\alpha}^T \int \tilde{H}^2_x\tilde{\mu}(1-\tilde{\mu})dxdt \\
&=\; B_1 + B_2 + B_3 + B_4.
\end{aligned}$$

Choose $L = L(\tilde{H}_x)$ large so that $B_2 \le \epsilon/4$, and take $m = m(\tilde{H}_x, L)$ large enough so that both $B_1, B_3 \le \epsilon/4$.

The term $B_4/2$ is the rate of $\tilde{\mu}$ on the time interval $[T-\alpha, T]$. By the construction of $\tilde{\mu}$, convexity of the the rate, and translation-invariance, we estimate

$$\begin{aligned}
B_4 \;&\le\; 2b\int \sigma_\alpha(z)\int_0^T \kappa_\varepsilon(s)\int_{T-\alpha}^T \int H^2_x\mu(1-\mu)(t+s-\beta, x-z)dtdxdsdz \\
&\le\; 2\int_{T-2\alpha}^T\int H^2_x\mu(1-\mu)dxdt,
\end{aligned}$$

taking $\beta \le \alpha$. Then, $B_4 \downarrow 0$ as $\alpha \downarrow 0$.

Hence, with α (and other parameters $1-b, \beta, \varepsilon$) small enough, there is M so that for $m \ge M$, $|I_0(\bar{\chi}) - I_0(w^m)| < \epsilon$, and also, by Lemma 5.5, $I_0(w^m) \le I_0(\mu) + 1$. Then, also with a perhaps smaller choice of α, by Lemma 5.1, we can arrange the Skorohod distance $d(\bar{\chi}; w^m) < \epsilon$ for $m \ge M$. □

Proof of Proposition 1.3. Let γ be a profile associated to an (LEM) or (DIC) measure, and let μ be such that $I_\gamma(\mu) < \infty$. By successively applying Lemmas 5.2, 5.3, 5.4, 5.5 and 5.6, we can build an approximating sequence μ^n to verify

$\mu \in \mathcal{A}$. Specifically, under an (LEM) initial measure, when $I_\gamma(\mu) = I_\gamma^{LE}(\mu)$, the sequence μ^n is built from w^m in Lemma 5.5 with appropriate choice of parameters $b, \alpha, \beta, \varepsilon$ and m. Under a (DIC) initial configuration, when $I_\gamma(\mu) = I_\gamma^{DC}(\mu)$, μ^n is constructed from $\bar{\chi}$ in Lemma 5.6 again with a choice of parameters. □

Proof of Proposition 2.2. The proposition follows by applying Lemmas 5.2 and 5.3 to build a sequence $\{\mu^n\}$ which satisfies specifications (i), (ii), (iii), (iv), (v), (vi), and (vii). Property (viii) also follows: When $\mu_0 = \rho$, by construction, noting (5.1) with $\gamma = \rho$, $\tilde{\mu}_0 = \hat{\mu}_0 = \mu^{b,\alpha} = \rho$. □

5.2. **Proof of Lemmas 2.3, 2.4.** We begin with a lemma.

Lemma 5.7. *Let μ be a smooth density satisfying (1.3) such that $\|\mu_0 - \rho\|_{L^\infty(\mathbb{R})} < A_0$, and $I_0(\mu) < A_1$. Then, in terms of a universal constant C, for all $x \in \mathbb{R}$ and $0 < \epsilon < 1$,*

$$\sup_{t\in[0,T]} \int_x^{x+\epsilon} (\mu(t,z) - \rho)dz \ \le\ A_0 + CT^{1/4}\sqrt{A_1}.$$

Proof. First note that

$$\int_x^{x+\epsilon} (\mu(t,y) - \rho)dy \ =\ \lim_{h\downarrow 0} \int_x^{x+\epsilon} \sigma_h * (\mu - \rho)(t,y)dy.$$

Now, write (cf. (5.3)) that

$$\int_x^{x+\epsilon} \sigma_h * (\mu(t,y) - \rho)dy \ \le\ \int_x^{x+\epsilon} \sigma_{t+h} * |\mu(0,y) - \rho|dy$$
$$+\Big|\int_x^{x+\epsilon}\int_0^t \int H_x\mu(1-\mu)(s,z)\partial_z\sigma_{t-s+h}(z-y)dzdsdy\Big|.$$

The first term can be bounded by A_0 uniformly in $x \in \mathbb{R}$ and $h > 0$.

The second term is bounded by

$$\int_0^t \int \big(|H_x|\mu(1-\mu)\big)(s,z)[\sigma_{t-s+h}(z-x) + \sigma_{t-s+h}(z-x-\epsilon)]dzds.$$

Since $\sigma_{t-s+h}(z-\alpha)$ has $L^2([0,T]\times\mathbb{R})$ bound $CT^{1/4}$ uniformly in $\alpha \in \mathbb{R}$ and $h > 0$, and the second term can be dominated by $CT^{1/4}\sqrt{A_1}$ as desired. □

Proof of Lemma 2.3. Following the first part of the argument for Lemma 5.7, we have

$$\int_x^{x+\epsilon} \mu(t,y) - \gamma(y)dy$$
$$\le\ \lim_{h\downarrow 0}\Big|\int_x^{x+\epsilon}\int \sigma_{t+h}(y-z)(\mu_0(z) - \gamma(z))dzdy\Big|$$
$$+\lim_{h\downarrow 0}\Big|\int_x^{x+\epsilon}\int_0^t\int H_x\mu(1-\mu)(s,z)\partial_z\sigma_{t-s+h}(z-y)dzdsdy\Big|.$$

The second integral is estimated as in Lemma 5.7, and bounded by

$$CT^{1/4}e^{-l^2/8T}\sqrt{I_0(\mu)} + CT^{1/4}\left[\int_0^T\int_{-l+x}^{l+x} H_x^2\mu(1-\mu)dzds\right]^{1/2}$$

which vanishes as $|x| \uparrow \infty$, and then $l \uparrow \infty$. The first integral is bounded by

$$\begin{aligned}
&\int dz\sigma_t(z)\int_x^{x+\epsilon} dy(\mu_0(y-z)-\gamma(y-z)) \\
&\quad \leq\ 2\sqrt{\epsilon}\int dz\sigma_t(z)\left[\int_x^{x+\epsilon}(\sqrt{\mu_0(y-z)}-\sqrt{\gamma(y-z)})^2 dy\right]^{1/2}
\end{aligned}$$

using Schwarz inequality. From the Hellinger inequality, $(\sqrt{a}-\sqrt{b})^2 \leq h_d(a;b)$, we obtain the further estimate on the right-side as

$$\begin{aligned}
&2\sqrt{\epsilon}\int dz\sigma_t(z)\left[\int_{x-z}^{x-z+\epsilon} h_d(\mu_0(y);\gamma(y))dy\right]^{1/2} \\
&\quad \leq 2\sqrt{\epsilon}\sup_{|z|\leq l}\left[\int_{x-z}^{x-z+\epsilon} h_d(\mu_0(y);\gamma(y))dy\right]^{1/2}\int_{|z|\leq l}\sigma_t(z)dz \\
&\qquad +C\sqrt{\epsilon}e^{-l^2/2T}h^{1/2}(\mu_0;\gamma)\int_{|z|\geq l}\sigma_{2t}(z)dz \\
&\quad \leq C\sqrt{\epsilon}\left[\sup_{|z|\leq l}\left[\int_{x-z}^{x-z+\epsilon} h_d(\mu_0(y);\gamma(y))dy\right]^{1/2} + e^{-l^2/2T}h^{1/2}(\mu_0;\gamma)\right]
\end{aligned}$$

which vanishes uniformly over $t\in[0,T]$ as $x\uparrow\infty$ and then $l\uparrow\infty$.

Now,

$$\begin{aligned}
\left|\int_x^{x+\epsilon}\mu(t,y)-\mu(t,x)dy\right| &= \left|\int_x^{x+\epsilon}\int_x^y \partial_x\mu(t,z)dzdy\right| \\
&\leq \epsilon^2\|\partial_x\mu\|_{L^\infty([0,T]\times\mathbb{R})}.
\end{aligned}$$

Then, noting $\gamma\in M_1(\rho_*,\rho^*)$,

$$\begin{aligned}
&\lim_{|x|\uparrow\infty}\sup_{t\in[0,T]}|\mu_t(x)-\gamma(x)| \\
&\quad \leq \lim_{|x|\uparrow\infty}\sup_{t\in[0,T]}\frac{1}{\epsilon}\left|\int_x^{x+\epsilon}(\mu_t(y)-\gamma(y))-(\mu_t(x)-\gamma(x))\,dy\right| \\
&\qquad +\lim_{|x|\uparrow\infty}\sup_{t\in[0,T]}\frac{1}{\epsilon}\left|\int_x^{x+\epsilon}\mu_t(y)-\gamma(y)dy\right| \\
&\quad \leq \epsilon\|\partial_x\mu\|_{L^\infty}.
\end{aligned}$$

As $0<\epsilon<1$ is arbitrary, the claim follows. □

Proof of Lemma 2.4. First, as $\mu_0^n(x)\equiv\rho$, from Lemma 5.7, for $(t,x)\in[0,T]\times\mathbb{R}$ and fixed $0<\epsilon<1$,

$$\int_x^{x+\epsilon}\big(\mu^n(t,y)-\rho\big)dy \quad\leq\quad CT^{1/4}\sqrt{I_0(\mu^n)}$$

which vanishes as $n\uparrow\infty$.

Also, given $\lim_{n\to\infty}\|\partial_x\mu^n\|_{L^2([0,T]\times\mathbb{R})}=0$, for fixed $\epsilon>0$, we have

$$\begin{aligned}\mu^n(t,x)-\rho &= \frac{1}{\epsilon^2}\int_t^{t+\epsilon}\int_x^{x+\epsilon}\mu^n(s,y)-\mu^n(t,x)dyds+O(1)\\ &= \frac{1}{\epsilon^2}\int_t^{t+\epsilon}\int_x^{x+\epsilon}\int_x^y\partial_x\mu^n(s,z)dzdyds+O(1)\\ &\leq \|\partial_x\mu^n\|_{L^2}+O(1)\ \to\ 0.\end{aligned}$$

Hence, we conclude that $\lim_{n\uparrow\infty}\|\mu^n(t,x)-\rho\|_\infty=0$. □

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Department of Mathematics, 396 Carver Hall, Iowa State University, Ames, IA 50011
e-mail: sethuram@iastate.edu

Courant Institute, 212 Mercer St., New York, NY 10012
e-mail: texttvaradhan@cims.nyu.edu