

## A REMARK ON THE MAXIMUM EIGENVALUE FOR CIRCULANT MATRICES

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ABSTRACT. We point out that the method of Davis-Mikosch [1] gives for a symmetric circulant  $n \times n$  matrix composed of i.i.d. entries with mean 0 and finite  $(2 + \delta)$ -moments in the first half-row that the maximum eigenvalue is on the order  $\sqrt{2n \log n}$ , and the fluctuations are Gumbel.

Let  $\{X_0, X_1, \dots\}$  be i.i.d. mean-zero, variance 1, random variables. For  $m \geq 1$ , consider the  $(2m + 1) \times (2m + 1)$  “palindromic” circulant matrix

$$\begin{bmatrix} X_0 & X_1 & X_2 & \cdots & X_m & X_m & X_{m-1} & \cdots & X_1 \\ \vdots & & & & \vdots & & & & \vdots \\ X_m & X_{m-1} & \cdots & & X_0 & X_1 & X_2 & \cdots & X_m \\ \vdots & & & & \vdots & & & & \vdots \\ X_1 & X_2 & X_3 & \cdots & X_m & X_{m-1} & X_{m-2} & \cdots & X_0 \end{bmatrix}. \quad (1)$$

In this note, we observe, for circulant matrices (1), that an argument of [1] for the maximum of periodograms easily applies to deduce that the maximum eigenvalue is on the order  $\sqrt{2m \log m}$ , and the fluctuations are Gumbel (Theorem 1). In particular, a sort of “universality” with respect to the entries  $\{X_i\}$ , much discussed in other contexts in the random matrix literature, is established for the asymptotic maximum eigenvalue distribution. We refer to [3] for more discussion of random circulant matrices, and note the result for Gaussian entries is as well given in [3, Corollary 5].

**Theorem 1.** *Suppose  $X_1, X_2, \dots$  are i.i.d. with  $E(X_1) = 0$ ,  $E(X_1^2) = 1$ , and  $E(|X_1|^s) < \infty$  for some  $s > 2$ . Denote by  $\lambda_m$  the maximum eigenvalue of (1), and let  $a_m = \sqrt{2 \log m} - \log(4\pi \log m)/(2\sqrt{2 \log m})$ . Then*

$$\lim_{m \rightarrow \infty} P\left(\left(\frac{\lambda_m}{\sqrt{2m+1}} - a_m\right) \sqrt{2 \log m} \leq x\right) = G(x)$$

where  $G(x) = \exp(-e^{-x})$ .

The proof follows closely the method used to prove [1, Theorem 2.1] which is based on Einmahl’s multivariate extension of the Komlos-Major-Tusnady theorem (cf. Lemma 3). Indeed, Lemmas 4, 5 are similar to [1, Lemmas 3.3, 3.4] with analogous proofs. The well known Bonferroni inequalities (Lemma 2) and Lemma 3 are stated as [1, Lemmas 3.1, 3.2].

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**Lemma 2.** *Let  $A_1, \dots, A_n$  be measurable events. Then for every  $1 \leq k \leq \lfloor n/2 \rfloor$ ,*

$$\sum_{d=1}^{2k} (-1)^{d-1} S_d \leq P(A_1 \cup \dots \cup A_n) \leq \sum_{d=1}^{2k-1} (-1)^{d-1} S_d,$$

where  $S_d = \sum_{1 \leq j_1 < \dots < j_d \leq n} P(A_{j_1} \cap \dots \cap A_{j_d})$ .

The next statement is Einmahl's Corollary 1(b), page 31, in combination with the Remark on page 32 [2].

**Lemma 3.** *Let  $\xi_1, \dots, \xi_n$  be independent random vectors in  $\mathbb{R}^d$ . Assume that the moment generating function of  $\{\xi_i\}$  exists in a neighborhood of the origin, and that*

$$\text{cov}(\xi_1 + \dots + \xi_n) = B_n I_d,$$

where  $B_n > 0$  and  $I_d$  is the  $d$  dimensional identity matrix. Let  $\eta_k$  be independent  $N(0, \sigma^2 \text{cov}(\xi_k))$  random vectors for  $1 \leq k \leq n$  independent of  $\{\xi_i\}$ , and  $0 < \sigma^2 \leq 1$ . Let  $\xi_k^* = \xi_k + \eta_k$  for  $1 \leq k \leq n$ , and write  $p_n^*$  as the density of  $B_n^{-1/2} \sum_{k=1}^n \xi_k^*$ . Choose  $0 < \alpha < 1/2$  such that

$$\alpha \sum_{k=1}^n E|\xi_k|^3 \exp(\alpha|\xi_k|) \leq B_n. \quad (2)$$

Let

$$\beta_n = \beta_n(\alpha) = B_n^{-3/2} \sum_{k=1}^n E|\xi_k|^3 \exp(\alpha|\xi_k|). \quad (3)$$

If

$$|x| \leq c_1 \alpha B_n^{1/2}, \quad \sigma^2 \geq -c_2 \beta_n^2 \log \beta_n \quad \text{and} \quad B_n \geq c_3 \alpha^{-2}, \quad (4)$$

where  $c_1, c_2, c_3$  are constants depending only on  $d$ , then

$$p_n^*(x) = \phi_{(1+\sigma^2)I_d}(x) \exp(\bar{T}_n(x)) \quad \text{with} \quad |\bar{T}_n(x)| \leq c_4 \beta_n (|x|^3 + 1), \quad (5)$$

where  $\phi_C$  is the density of the  $d$ -dimensional centered Gaussian vector with covariance matrix  $C$  and  $c_4$  is a constant depending only on  $d$ .

Let now  $\{X_j\}_{j \geq 0}$  be as in Theorem 1. For  $j, m \geq 0$ , define  $\bar{X}_j = \bar{X}_j^{(m)} = X_j 1_{|X_j| \leq m^{1/s}} - E(X_j 1_{|X_j| \leq m^{1/s}})$ .

**Lemma 4.** *We have a.s. that*

$$\begin{aligned} & \frac{2}{\sqrt{2m+1}} \max_{1 \leq j \leq m} \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) X_k \\ & - \frac{2}{\sqrt{2m+1}} \max_{1 \leq j \leq m} \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) \bar{X}_k^{(m)} = O(m^{-1/2}). \end{aligned}$$

*Proof.* First, we can add and subtract  $(2m+1)^{-1/2} \bar{X}_0$  on the left-side. Since

$$1 + 2 \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) = 0,$$

we can replace

$$\bar{X}_0 + 2 \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) \bar{X}_k$$

with

$$X_0 1_{|X_0| \leq m^{1/s}} + 2 \sum_{k=1}^m \cos\left(\frac{2\pi j k}{2m+1}\right) X_k 1_{|X_k| \leq m^{1/s}}.$$

Now, by Borel-Cantelli, as  $\sum_t P(|X_t| > t^{1/s}) < \infty$ , we have  $|X_t| \leq t^{1/s}$  for all  $t \geq N(\omega)$  a.s. Then,

$$\begin{aligned} \sum_{t=1}^m |X_t - X_t 1_{|X_t| \leq m^{1/s}}| &= \sum_{t=1}^m |X_t| 1_{|X_t| > m^{1/s}} \\ &\leq \sum_{t=1}^{N(\omega)} X_t 1_{|X_t| > m^{1/s}} + \sum_{t=N(\omega)+1}^m X_t 1_{|X_t| > t^{1/s}} \\ &\leq \sum_{t=1}^{N(\omega)} |X_t| 1_{|X_t| > m^{1/s}} + \sum_{t > N(\omega)} X_t 1_{|X_t| > t^{1/s}} = 0 \end{aligned}$$

for  $m \geq \max\{N(\omega), |X_1|^s, \dots, |X_{N(\omega)}|^s\}$ . Hence, the sums

$$\sum_{k=1}^m \cos\left(\frac{2\pi j k}{2m+1}\right) X_k \quad \text{and} \quad \sum_{k=1}^m \cos\left(\frac{2\pi j k}{2m+1}\right) X_k 1_{|X_k| \leq m^{1/s}}$$

agree for all large  $m$  a.s.

We finish by noting the extra term

$$\frac{1}{\sqrt{2m+1}} [\bar{X}_0 - X_0 1_{|X_0| \leq m^{1/s}}] = \frac{1}{\sqrt{2m+1}} E[X_0 1_{|X_0| \leq m^{1/s}}] = O(m^{-1/2}).$$

□

For  $d \geq 1$ , define  $v_d(t) = \langle \cos\left(\frac{2\pi j_1 t}{2m+1}\right), \dots, \cos\left(\frac{2\pi j_d t}{2m+1}\right) \rangle$  with respect to distinct integers  $1 \leq j_1, \dots, j_d \leq m$ . Let also  $\{N_j\}$  be a sequence of i.i.d.  $N(0, 1)$  random variables independent of  $\{X_j\}$ .

**Lemma 5.** For  $d \geq 1$ , let  $\tilde{p}_m$  be the density of

$$\frac{1}{\sqrt{E[\bar{X}_1^2](2m+1)}} \left[ \sqrt{2}(\bar{X}_0 + \sigma_m N_0) v_d(0) + 2 \sum_{k=1}^m (\bar{X}_k + \sigma_m N_k) v_d(k) \right]$$

where  $\sigma_m^2 = E[\bar{X}_1^2] s_m^2$ . If  $m^{-2c_5} \log m \leq s_m^2 \leq 1$  for  $c_5 = 1/2 - (1-\delta)/s > 0$  and some  $0 < \delta < 1$ , then, uniformly for  $|x|^3 = o(m^{1/2-1/s})$ ,

$$\tilde{p}_m(x) = \phi_{(1+s_m)I_d}(x)(1 + o(1)).$$

*Proof.* Apply Lemma 3 to the centered vectors  $\sqrt{2}\bar{X}_0 v_d(0), 2\bar{X}_1 v_d(1), \dots, 2\bar{X}_m v_d(m)$  where, after some calculation,

$$\text{cov}\left(\sqrt{2}\bar{X}_0 v_d(0) + 2\bar{X}_1 v_d(1) + \dots + 2\bar{X}_m v_d(m)\right) = B_m I_d$$

and

$$B_m = E\bar{X}_1^2 \left[ 2 + 4 \sum_{k=1}^m \cos^2\left(\frac{2\pi k}{2m+1}\right) \right] = (2m+1) E\bar{X}_1^2.$$

Choose for a fixed constant  $c_6 > 0$ ,

$$\tilde{\alpha} = c_6 m^{-1/s} d^{-1/2}.$$

Note for each  $0 \leq t \leq m$ , that

$$|v_d(t)|^2 = \sum_{l=1}^d \cos^2\left(\frac{2\pi jlt}{2m+1}\right) \leq d.$$

Then, for large  $m$ ,

$$\begin{aligned} & \tilde{\alpha} E|\sqrt{2}\bar{X}_0 v_d(0)|^3 \exp\{\tilde{\alpha}\sqrt{2}\bar{X}_0 v_d(0)\} + \tilde{\alpha} \sum_{t=1}^m E|2\bar{X}_t v_d(t)|^3 \exp\{\tilde{\alpha}|2\bar{X}_t v_d(t)|\} \\ & \leq 8d^{3/2}\tilde{\alpha}(m+1)E|\bar{X}_1|^3 \exp\{2\tilde{\alpha}|\bar{X}_1|d^{1/2}\} \\ & \leq 10dc_6 m^{1-1/s} E|\bar{X}_1|^3 \exp\{2c_6\} \\ & \leq 10dc_6 \exp\{4c_6\} m^{1-\delta/s} E|X_1|^{2+\delta} \end{aligned}$$

where  $0 < \delta < 1$  is chosen so that  $E|X_1|^{2+\delta} < \infty$ . Then, (2) holds with  $\alpha = \tilde{\alpha}$  for sufficiently small  $c_6$ .

Now choose

$$\begin{aligned} \tilde{\beta}_m & = B_m^{-3/2} E|\sqrt{2}\bar{X}_0 v_d(0)|^3 \exp\{\tilde{\alpha}\sqrt{2}\bar{X}_0 v_d(0)\} \\ & \quad + B_m^{-3/2} \sum_{t=1}^m E|2\bar{X}_t v_d(t)|^3 \exp\{\tilde{\alpha}|2\bar{X}_t v_d(t)|\} \\ & \leq 8d^{3/2} B_m^{-3/2} (m+1) E|\bar{X}_1|^3 \exp\{2\tilde{\alpha}|\bar{X}_1|d^{1/2}\}. \end{aligned}$$

Then,

$$\tilde{\beta}_m \leq \text{const}(B_m^{-3/2} m^{1+(1-\delta)/s} E|\bar{X}_1|^{2+\delta}) \leq \text{const}(m^{-c_5})$$

where  $c_5 = 1/2 - (1-\delta)/s > 0$ .

Next, we consider (4). We can choose  $x$  so that

$$|x| \leq c_1 \tilde{\alpha} B_m^{1/2} \sim \text{const}(m^{1/2-1/s}).$$

Then, we can choose  $\sigma^2 = s_m^2$  so that

$$1 \geq s_m^2 \geq \text{const}(m^{-2c_5} \log m)$$

and note

$$B_m \sim m \geq c_3 \tilde{\alpha}^{-2} \sim m^{2/s}.$$

Noting (5), we have

$$\tilde{p}_m = \phi_{(1+s_m^2)I_d}(x) \exp(\bar{T}_m(x)) \quad \text{with} \quad |\bar{T}_m(x)| \leq c_4 \tilde{\beta}_m (|x|^3 + 1).$$

However, uniformly over  $|x|^3 = o(m^{1/2-1/s})$ ,

$$|\bar{T}_m(x)| \leq c_4 \tilde{\beta}_m (|x|^3 + 1) \leq \text{const}(m^{1/2-1/s-c_5}) = \text{const}(m^{-\delta/s}) = o(1).$$

□

**Proof of Theorem 1.** From properties of circulants, we have that the eigenvalues of (1) are  $\lambda_j = X_0 + 2 \sum_{k=1}^m \cos\left(\frac{2\pi kj}{2m+1}\right) X_k$  for  $0 \leq j \leq 2m$ , and also  $\lambda_j = \lambda_{2m+1-j}$  for  $1 \leq j \leq m$ . Since  $m^{-1/2} \sqrt{2 \log m} \rightarrow 0$ , the variable  $X_0$  in the expression for  $\lambda_j$  can be replaced by  $\sqrt{2}\bar{X}_0$ . We will also be able to omit the contribution of  $\lambda_0$  to the maximum. By Lemma 4, it will be enough to prove

$$\sqrt{2 \log m} \left[ \frac{\sqrt{2}\bar{X}_0}{\sqrt{2m+1}} + \max_{1 \leq j \leq m} \frac{2}{\sqrt{2m+1}} \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) \bar{X}_k - a_m \right] \Rightarrow G. \quad (6)$$

To this end, let  $\sigma_m^2 = E[\bar{X}_1^2]s_m^2 = E[\bar{X}_1^2]m^{-2c_5} \log m$ . We first show

$$\begin{aligned} & \sqrt{2 \log m} \left[ \frac{\sqrt{2}(\bar{X}_0 + \sigma_m N_0)}{\sqrt{E[\bar{X}_1^2](2m+1)}} \right. \\ & \quad \left. + \max_{1 \leq j \leq m} \frac{2}{\sqrt{E[\bar{X}_1^2](2m+1)}} \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) (\bar{X}_k + \sigma_m N_k) - a_m \right] \Rightarrow G. \end{aligned} \quad (7)$$

For  $1 \leq j \leq m$ , let

$$\lambda_j^{\bar{X}+N} = \frac{1}{\sqrt{E[\bar{X}_1^2](2m+1)}} \left[ \sqrt{2}(\bar{X}_0 + \sigma_m N_0) + 2 \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) (\bar{X}_k + \sigma_m N_k) \right].$$

Since  $1 - e^{-e^{-u}} = \sum_{d=1}^{\infty} (-1)^{d-1} (e^{-du}/d!)$ , by Lemma 2, (7) will follow from the statement

$$\begin{aligned} & P\left(\lambda_{j_1}^{\bar{X}+N} > a_m + \frac{u}{\sqrt{2 \log m}}, \dots, \lambda_{j_d}^{\bar{X}+N} > a_m + \frac{u}{\sqrt{2 \log m}}\right) \\ & \quad = m^{-d} \exp(-du)(1 + o(1)) \end{aligned} \quad (8)$$

uniformly over the  $d$ -tuples  $1 \leq j_1 < \dots < j_d \leq m$  for each  $d \geq 1$  as  $m \uparrow \infty$ .

Let  $A_m^d$  denote the event in the probability on the left-side. Then, noting  $s_m^2 = m^{-2c_5} \log m$ ,

$$\int_{A_m^d} \phi_{(1+s_m^2)I_d}(x) dx = m^{-d} \exp\{-du\}(1 + o(1))$$

as  $m \uparrow \infty$ . Note that we can neglect the parts in (8) when there is  $l \leq d$  such that

$$|\lambda_{j_l}^{\bar{X}+N}|^3 > m^{1/2-1/s-\epsilon}$$

for a small  $\epsilon > 0$ . Indeed, given  $d \geq 1$  and  $s > 2$  choose  $0 < \epsilon < 1/2 - 1/s$  and  $\gamma > 2$  such that  $\gamma(1/2 - 1/s - \epsilon) > d + 1$ . Note also  $1/2 \leq E[\bar{X}_1^2] \leq 2$  for  $m$  large enough. Then, by Rosenthal's inequality there is a constant  $C(\gamma)$  such that

$$\begin{aligned} & P\left(\left|\sqrt{2}\bar{X}_0 + 2 \sum_{k=1}^m \cos\left(\frac{2\pi j_l k}{2m+1}\right) \bar{X}_k\right|^3 > m^{2-1/s-\epsilon}\right) \\ & \leq \frac{C(\gamma)}{m^{\gamma(2-1/s-\epsilon)}} \left( \left(\sum_{k=0}^m E|\bar{X}_k|^2\right)^{3\gamma/2} + mE|\bar{X}_1|^{3\gamma} \right) \\ & \leq C(\gamma) \left( \frac{2}{m^{\gamma(1/2-1/s-\epsilon)}} + \frac{m}{m^{\gamma(2-4/s-\epsilon)}} \right) = o(m^{-d}). \end{aligned}$$

On the other hand, also

$$P\left(\left|\frac{\sqrt{2}N_0}{\sqrt{2m+1}} + \frac{2}{\sqrt{2m+1}} \sum_{k=1}^m \cos\left(\frac{2\pi j_l k}{2m+1}\right) \sigma_m N_k\right|^3 > m^{1/2-1/s-\epsilon}\right) = o(m^{-d}).$$

We conclude by Lemma 5 (which does not depend on the choice of  $j_1, \dots, j_d$ ) that (8) holds.

To deduce (6), note  $E[\bar{X}_1^2] \rightarrow E[X_1^2] = 1$ , and

$$\begin{aligned}
\frac{\sqrt{2}\sigma_m N_0}{\sqrt{2m+1}} - \max_{1 \leq j \leq m} \frac{2}{\sqrt{2m+1}} \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) (-\sigma_m N_k) \\
\leq \frac{\sqrt{2}(\bar{X}_0 + \sigma_m N_0)}{\sqrt{2m+1}} + \max_{1 \leq j \leq m} \frac{2}{\sqrt{2m+1}} \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) (\bar{X}_k + \sigma_m N_k) \\
- \frac{\sqrt{2}\bar{X}_0}{\sqrt{2m+1}} - \max_{1 \leq j \leq m} \frac{2}{\sqrt{2m+1}} \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) \bar{X}_k \\
\leq \frac{\sqrt{2}\sigma_m N_0}{\sqrt{2m+1}} + \max_{1 \leq j \leq m} \frac{2}{\sqrt{2m+1}} \sum_{k=1}^m \cos\left(\frac{2\pi jk}{2m+1}\right) \sigma_m N_k. \tag{9}
\end{aligned}$$

Let  $(2m+1)^{1/2}\lambda_j^N = \sqrt{2}\sigma_m N_0 + 2 \sum_{k=1}^m \cos(2\pi jk/2m+1)\sigma_m N_k$  for  $1 \leq j \leq m$ . One can calculate that that  $\{\lambda_j^N\}_{j=1}^m$  are i.i.d.  $N(0, \sigma_m^2)$  variables.

Hence, to finish, the bounds (9) correspond to the maximum of  $m$  i.i.d.  $N(0, \sigma_m^2)$  random variables, well known to be on order  $\sigma_m \sqrt{2 \log m} \sim m^{-c_5} \log m \rightarrow 0$  in probability.  $\square$

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