Tiling on multipartite graphs

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Texas State Discrete Math Seminar
1 Hajnal-Szemerédi
2 Multipartite graphs
3 Extremal examples
4 Multipartite factors
5 Approximate bounds
6 Critical chromatic number
7 Open problems

This talk includes joint work with:

- Csaba Magyar
- Endre Szemerédi, Rutgers University and the Rényi Institute
- Yi Zhao, Georgia State University
The Hajnal-Szemerédi theorem

Theorem (Hajnal-Szemerédi, 1970)

(Complementary form) If \( G \) is a simple graph on \( n \) vertices with minimum degree

\[
\delta(G) \geq \left(1 - \frac{1}{r}\right)n
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then \( G \) contains a subgraph which consists of \( \lfloor n/r \rfloor \) vertex-disjoint copies of \( K_r \).
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- $r = 2$ follows from Dirac
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- New proof by Kierstead & Kostochka 2008 (discharging)
The Alon-Yuster theorem

Theorem (Alon-Yuster, 1992)

For any $\alpha > 0$ and graph $H$, there exists an $n_0 = n_0(\alpha, H)$ such that in any graph $G$ on $n \geq n_0$ vertices with

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)} + \alpha\right) n$$

there is an $H$-factor of $G$ if $|V(H)|$ divides $n$. 

Komlós, Sárközy and Szemerédi, 2001, showed that $n$ can be replaced by $C = C(H)$, but not eliminated entirely.
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Multipartite graphs

Definition

The family of $r$-partite graphs with $N$ vertices in each part is denoted $G_r(N)$. Note that $G \in G_r(N) \Rightarrow |V(G)| = rN$. 

Definition

The natural bipartite subgraphs of $G$ are the ones induced by the pairs of classes of the $r$-partition.
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The natural bipartite subgraphs:
Minimum degree condition

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If $G \in \mathcal{G}_r(N)$, let $\bar{\delta}(G)$ denote the minimum degree among all of the natural bipartite subgraphs of $G$. 

Conjecture [Fischer]

If $G \in \mathcal{G}_r(N)$ and $\bar{\delta}(G) \geq 1 - \frac{1}{r}N$ then $G$ has a $K_r$-factor.
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I.e., each vertex $v \in V_1$ has at least $\bar{\delta}(G)$ neighbors in each of $V_2, V_3, \ldots, V_r$. 
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Let $G \in G_r(N)$ and $\bar{\delta}(G) \geq (1 - \frac{1}{r}) N$. Then,

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So the total degree is not large enough to invoke Hajnal-Szemerédi.
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$$= \left(1 - \frac{2r-1}{r^2}\right) |V(G)|$$

$$< \left(1 - \frac{1}{r}\right) |V(G)|, \text{ if } r \geq 2.$$  

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The bound $\delta(G) \geq (1 - \frac{1}{r}) N$ is not sufficient for $(r, N)$ such that $r$ is odd, and $N$ is an odd multiple of $r$. 

Example

Let $r = 3$ and $N = 3$: 
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The bound $\bar{\delta}(G) \geq (1 - \frac{1}{r}) N$ is not sufficient for $(r, N)$ such that

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Example Let $r = 3$ and $N = 3$:
General Example

Redraw the example with nonedges:

\[ \Gamma_3(3) \]
Redraw the example with nonedges:

This complement can be attributed to Paul Catlin, 1976, and was called a “type 2 graph.”
For any $N$, with $r \mid N$, we can "blow up" this graph by $N/r$: 

- Replace each vertex with $N/r$ vertices.
- Replace each edge with $K_{N/r}$, $N/r$.
Blowing up

For any $N$, with $r | N$, we can ”blow up” this graph by $N/r$:

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Blowing up

For any $N$, with $r \mid N$, we can ”blow up” this graph by $N/r$:

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Then, $\Gamma_r(N) \in \mathcal{G}_r(N)$.
Blowing up

For any \( N \), with \( r \mid N \), we can ”blow up” this graph by \( N/r \):

- Replace each vertex with \( N/r \) vertices.
- Replace each edge with \( K_{N/r,N/r} \).

Then, \( \Gamma_r(N) \in \mathcal{G}_r(N) \).

If \( r \mid N \), then \( \Gamma_r(N/r) \) has no \( K_r \)-factor iff \( r \) is odd and \( N/r \) is odd.
Theorem (Magyar-M, 2002)

There exists an $N_0$ such that if $N \geq N_0$, $G \in \mathcal{G}_3(N)$ and

$$\bar{\delta}(G) \geq \frac{2}{3}N,$$

then $G$ has a $K_3$-factor unless

$G \approx \Gamma_3(N)$ and $N/3$ is an odd integer.
Theorem (Magyar-M, 2002)

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($N$ need not be divisible by 3.)
Quadripartite theorem

Theorem (M-Szemerédi, 2008)

There exists an $N_0$ such that if $N \geq N_0$, $G \in G_4(N)$ and

$$\bar{\delta}(G) \geq \frac{3}{4} N,$$

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Quadripartite theorem

**Theorem (M-Szemerédi, 2008)**

There exists an $N_0$ such that if $N \geq N_0$, $G \in \mathcal{G}_4(N)$ and

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There is no exceptional graph.
Bipartite graph factors

**Theorem (Zhao, 2009)**

Let \( h \) be a positive integer. There exists an \( N_0 = N_0(h) \) such that if \( N \geq N_0, \ h \mid N, \) and \( G \in \mathcal{G}_2(N) \) with

\[
\bar{\delta}(G) \geq \begin{cases} 
\frac{N}{2} + h - 1, & N/h \text{ is odd;} \\
\frac{N}{2} + \frac{3h}{2} - 2, & N/h \text{ is even,}
\end{cases}
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then \( G \) has a \( K_{h,h} \)-factor.
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then $G$ has a $K_{h,h}$-factor.

Moreover, there are examples that prove that this $\bar{\delta}$ condition cannot be improved.
Two-colorable graph factors

Note

If $\chi(H) = 2$ and $|V(H)| = h$, then $K_{h,h}$-factor $\Rightarrow$ $H$-factor.
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Example.
Tripartite graph factors

Theorem (M-Zhao, 2009+)

Let $h$ be a positive integer and $f(h)$ be the minimum integer such that:

1. $\exists N_0 = N_0(h)$ for which $N \geq N_0$, $h \mid N$, $G \in G_3(N)$, $\bar{\ell}(G) \geq h^2 N^3 h + f(h)$ implies $G$ has a $K_{h,h,h}$-factor.

2. $f(h) = h - 1$, if $N/h \equiv 0 \mod 6$;
   $h - 2 \leq f(h) \leq h - 1$, if $N/h \not\equiv 0 \mod 3$;
   $h - 1 \leq f(h) \leq 2h - 1$, if $N/h \equiv 3 \mod 6$.

Note: Both $(H) = 3$ and $|V(H)| = h$ together imply a $H$-factor also.
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implies $G$ has a $K_{h,h,h}$-factor.

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Note

Both $\chi(H) = 3$ and $|V(H)| = h$ together imply a $H$-factor also.
Other graph factors

The case analysis required to prove that $\bar{\delta}(G) \geq (3/4 + \epsilon)N$ is sufficient for a $K_{h,h,h,h}$-factor would be long and difficult, using current methods. However, we believe it could be done.
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To prove the existence of an $f(h)$ would be even more difficult.
No version of the following key lemma for $r \geq 5$:

Almost-covering lemma ($r = 3$)

For every $\Delta > 0$, there exists an $\varepsilon > 0$ such that if $G \in G_3(N)$, $\overline{\Gamma}(G) \geq 2 - 3N$ and $T_0$ is a partial $K_3$-factor of $G$ with $|T_0| < N - 3$, then either

1. $\exists$ a partial $K_3$-factor $T'$ with $|T'| > |T_0|$ and $|T' \setminus T_0| \leq 15$ or
2. $\exists$ 3 sets which are each of size $N/3$ but have pairwise density $\leq \Delta$. 

Ryan Martin (Iowa State U.)
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Best general bound

**Theorem (Csaba-Mydlarz, 2009+)**

Let \( r \geq 5 \) and \( \epsilon > 0 \). There exists an \( N_0 = N_0(r, \epsilon) \) such that if \( N \geq N_0 \), \( G \in \mathcal{G}_r(N) \) and if

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\tilde{\delta}(G) \geq \left( \frac{k}{k + 1} + \epsilon \right) N, \quad k = r + \lceil 4h_r \rceil,
\]

then \( G \) has a \( K_r \)-factor.
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$$h_r = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r}$$
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h_r = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r}
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This is the best bound for \( r \geq 5 \).
Critical chromatic number

Definition

Let $H$ be a graph with

- order: $h = |V(H)|$
Critical chromatic number

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Let $H$ be a graph with

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- $\sigma$ is the size of the smallest color class of $H$ among all proper $\chi$-colorings of $V(H)$. 

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The **critical chromatic number of** $H$, $\chi_{cr}(H)$ is

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\chi_{cr}(H) = \frac{(\chi - 1)h}{h - \sigma}.
$$
Critical chromatic number

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**Fact**

For any graph $H$:

$$\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$$
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**Fact**
For any graph $H$:  

$$\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$$

Also, $\chi_{cr}(H) = \chi(H)$ iff every proper $\chi$-coloring of $H$ is a equipartition.
Critical chromatic number

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The critical chromatic number of $H$, $\chi_{cr}(H)$ is

$$\chi_{cr}(H) = \frac{(\chi-1)h}{h-\sigma}$$

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For any graph $H$:

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Also, $\chi_{cr}(H) = \chi(H)$ iff every proper $\chi$-coloring of $H$ is a equipartition.

$\chi_{cr}(H)$ was defined by Komlós, 2000.
Theorem (Komlós, 2000)

For every $H$ and every $n$, divisible by $|V(H)|$, there exists a $G$ of order $n$ with

$$\delta(G) = \left\lfloor \left( 1 - \frac{1}{\chi_{cr}(H)} \right)^n \right\rfloor - 1$$

and no $H$-factor.
Use of critical chromatic number

Theorem (Komlós, 2000)

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For every $H$ and $\epsilon > 0$, there exists $n_0 = n_0(H, \epsilon)$ such that if $G$ has order $n \geq n_0$ and

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then $G$ has an $H$-factor that covers all but $\epsilon n$ vertices in $G$. 
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**Theorem (Komlós, 2000)**

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Kühn & Osthus, 2009, gave a characterization of many $H$ for which

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right)n + C'$$

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**Question**

Does $\chi_{cr}$ provide a better minimum-degree parameter for finding an $H$-factor of an $r$-partite graph where $r = \chi(H)$?
Possible solution techniques

- Ideas from the Kierstead-Kostochka proof
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  - e.g., discharging
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- Ideas from the Csaba-Mydlarz proof
Possible solution techniques

- Ideas from the Kierstead-Kostochka proof
  - e.g., discharging

- Ideas from the Csaba-Mydlarz proof
  - there is a structure that might be modified to apply their main lemma.
Open problems

- Is it true that $\exists C$ such that $G \in \mathcal{G}_5(N)$ and $\bar{\delta}(G) \geq (4/5)N$ implies that there exists a partial $K_5$-factor of size $(1 - \epsilon)N$?
Open problems

- Is it true that $\exists C$ such that $G \in \mathcal{G}_5(N)$ and $\bar{\delta}(G) \geq (4/5)N$ implies that there exists a partial $K_5$-factor of size $(1 - \epsilon)N$?

- Is it true that, $\forall \epsilon > 0$, $\exists N_0 = N_0(\epsilon)$ such that $N \geq N_0$, $G \in \mathcal{G}_5(N)$ and $\bar{\delta}(G) \geq (4/5 + \epsilon)N$ implies a $K_5$-factor?
Open problems

- Is it true that $\exists C$ such that $G \in G_5(N)$ and $\bar{\delta}(G) \geq (4/5)N$ implies that there exists a partial $K_5$-factor of size $(1 - \epsilon)N$?

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Almost-covering question ($r = 5$)

Does there exist an absolute constant $C$ such that:

For all $\epsilon > 0$, if $G \in G_5(N)$,

$$\bar{\delta}(G) \geq \left(\frac{4}{5} + \epsilon\right)N$$

and $T_0$ is a partial $K_5$-factor of $G$ with $|T_0| < N - C$, then $\exists$ a partial $K_5$-factor $T'$ with $|T'| > |T_0|$?
Open problems

Almost-covering question \((r = 5)\)

Does there exist an absolute constant \(C\) such that:
For all \(\epsilon > 0\), if \(G \in \mathcal{G}_5(N)\),

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and \(T_0\) is a partial \(K_5\)-factor of \(G\) with \(|T_0| < N - C\), then \(\exists\) a partial \(K_5\)-factor \(T'\) with \(|T'| > |T_0|\)?

- Given a bipartite graph \(H\), what is the minimum degree required to ensure an \(H\)-factor in a bipartite graph, with appropriate divisibility conditions?
Open problems

Almost-covering question \((r = 5)\)

Does there exist an absolute constant \(C\) such that:
For all \(\epsilon > 0\), if \(G \in \mathcal{G}_5(N)\),

\[
\bar{\delta}(G) \geq \left(\frac{4}{5} + \epsilon\right)N
\]

and \(\mathcal{T}_0\) is a partial \(K_5\)-factor of \(G\) with \(|\mathcal{T}_0| < N - C\), then \(\exists\) a partial \(K_5\)-factor \(\mathcal{T}'\) with \(|\mathcal{T}'| > |\mathcal{T}_0|\)?

- Given a bipartite graph \(H\), what is the minimum degree required to ensure an \(H\)-factor in a bipartite graph, with appropriate divisibility conditions?

  I.e., \((1/2 + \epsilon)N\) is sufficient. What about \((1 - 1/\chi_{cr}(H) + \epsilon)N\)?
Bibliography


CsM09+ B. Csaba and M. Mydlarz, Approximate multipartite version of the Hajnal-Szemerédi theorem.

MZ09+ R. Martin and Y. Zhao, Tiling tripartite graphs with 3-colorable graphs, submitted.