An asymptotic multipartite Kühn-Osthus theorem

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Joint Work

This talk is based on joint work with:

Richard Mycroft
University of Birmingham

Jozef Skokan
London School of Economics

The Hajnal-Szemerédi theorem

Theorem (Hajnal-Szemerédi, 1970)

(Complementary form) If G is a simple graph on n vertices with minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{k}\right) n$$

then G contains a subgraph which consists of $\lfloor n/k \rfloor$ vertex-disjoint copies of K_k.

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- $k = 2$ follows from Dirac
- $k = 3$ proven by Corrádi & Hajnal 1963

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For any $\alpha > 0$ and graph H, there exists an $n_0 = n_0(\alpha, H)$ such that in any graph G on $n \geq n_0$ vertices with

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Komlós, Sárközy and Szemerédi, 2001, showed that αn can be replaced by $C = C(H)$, but not eliminated entirely.

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Theorem (Kühn-Osthus, 2009)

For any graph H*, there exists an* $n_0 = n_0(H)$ *and a constant* $C = C(H)$ *such that in any graph* G *on* $n \geq n_0$ *vertices with*

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This result is best possible, up to the constant C.

But what is $\chi^*(H)$?

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For any graph H:

$$\chi(H) - 1 < \chi_{\rm cr}(H) \leq \chi(H)$$

Also, $\chi_{\rm cr}(H) = \chi(H)$ iff every proper χ-coloring of H is a equipartition.

$\chi_{\rm cr}(H)$ was defined by Komlós, 2000.

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$$\chi^*(H) = \begin{cases} \chi_{\mathrm{cr}}(H), & \text{if } \mathrm{gcd}(H) = 1; \\ \chi(H), & \text{else.} \end{cases}$$

where $\mathrm{gcd}(H)$ is basically the gcd of the differences of the color classes in proper colorings of H.

Definitions

Definition

The family of k-partite graphs with n vertices in each part is denoted $\mathcal{G}_k(n)$.

Definition

The natural bipartite subgraphs of G are the ones induced by the pairs of classes of the k-partition.

Definition

If $G \in \mathcal{G}_k(n)$, let $\hat{\delta}_k(G)$ denote the minimum degree among all of the natural bipartite subgraphs of G.

Multipartite Hajnal-Szemerédi

The asymptotic Hajnal-Szemerédi theorem was solved with two different methods:

Theorem (Keevash-Mycroft, 2013; Lo-Markström, 2013)

Let $k \geq 2$ and $\epsilon > 0$. There exists an $n_0 = n_0(k, \epsilon)$ such that if $n \geq n_0$, $G \in \mathcal{G}_k(n)$ and if

$$\hat{\delta}_k(G) \geq \left(1 - \frac{1}{k}\right) n + \epsilon n,$$

then G has a K_k-tiling.

Hypergraph blow-up; Absorbing method

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Multipartite Hajnal-Szemerédi

In a longer manuscript, Keevash and Mycroft settle the multipartite Hajnal-Szemerédi case for large n:

Theorem (Keevash-Mycroft, 2013, *Mem. Amer. Math. Soc.*)

Let $k \geq 2$. There exists an $n_0 = n_0(k)$ such that if $n \geq n_0$, $G \in \mathcal{G}_k(n)$ and if

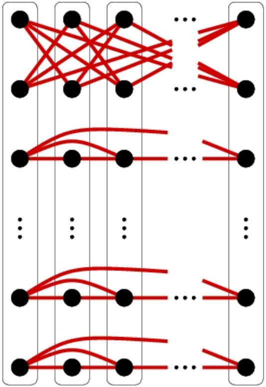
$$\hat{\delta}_k(G) \geq \left(1 - \frac{1}{k}\right) n,$$

then G has a K_k-tiling or both k and n/k are odd integers and $G \approx \Gamma_k(n/k)$.

The case of $k = 3$ was solved by Magyar-M. (2002). The case of $k = 4$ was solved by M.-Szemerédi (2008).

The graph $\Gamma_k(n/k)$ is one of Catlin's "Type 2" graphs.

Catlin's Type 2 Graphs



Catlin's Type 2 graph.

The red indicates non-edges between graph classes.

Toward Kühn-Osthus

Theorem (Zhao, 2009)

Let h be a positive integer. There exists an $n_0 = n_0(h)$ such that if $n \geq n_0$, $h \mid n$, and $G \in \mathcal{G}_2(n)$ with

$$\delta(G) = \hat{\delta}_2(G) \geq \begin{cases} \frac{1}{2}n + h - 1, & \text{if } n/h \text{ is odd;} \\ \frac{1}{2}n + \frac{3h}{2} - 2, & \text{if } n/h \text{ is even,} \end{cases}$$

then G has a perfect $K_{h,h}$-tiling.

Moreover, there are examples that prove that this $\hat{\delta}_2$ condition cannot be improved.

Toward Kühn-Osthus

Theorem (Bush-Zhao, 2012)

Let H be a bipartite graph. There exists an $n_0 = n_0(H)$ and $c = c(H)$ such that if $n \geq n_0$, $|V(H)| \mid n$, and $G \in \mathcal{G}_2(n)$ with

$$\delta(G) \geq \begin{cases} \left(1 - \frac{1}{\chi^*(H)}\right) n \quad +c, & \textit{if } \gcd(H) = 1 \textit{ or } \gcd_{\mathrm{cc}}(H) > 1; \\ \left(1 - \frac{1}{\chi(H)}\right) n \quad +c, & \textit{if } \gcd(H) > 1 \textit{ and } \gcd_{\mathrm{cc}}(H) = 1, \end{cases}$$

then G has a perfect H-tiling.

The quantity $\gcd_{\mathrm{cc}}(H)$ counts the gcd of the sizes of the connected components of H.

Our results

Theorem (M.-Skokan, 2013+)

Let $k \geq 2$, H be a graph with $\chi(H) = k$ and $\epsilon > 0$. There exists an $n_0 = n_0(H, \epsilon)$ such that if $n \geq n_0$, $G \in \mathcal{G}_k(n)$ and if

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This, of course, contains the asymptotic Hajnal-Szemerédi case.

Our results

Theorem (M.-Mycroft-Skokan, 2015+)

Let $k \geq 2$, H be a graph with $\chi(H) = k$, $\chi^* = \chi^*(H)$ and $\epsilon > 0$. There exists an $n_0 = n_0(H, \epsilon)$ such that if $n \geq n_0$, $G \in \mathcal{G}_k(n)$ and if

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The main tool is linear programming.

Linear programming

Definition

For any graph G, let $\mathcal{T}_k(G)$ denote the set of k-cliques of G. The FRACTIONAL K_k-TILING NUMBER, $\tau_k^*(G)$ *is:*

$$\tau_k^*(G) = \begin{cases} \max & \displaystyle\sum_{T \in \mathcal{T}_k(G)} w(T) \\ s.t. & \displaystyle\sum_{T \in \mathcal{T}_k(G), T \ni v} w(T) \leq 1, \quad \forall v \in V(G), \\ & w(T) \geq 0, \qquad\qquad \forall T \in \mathcal{T}_k(G). \end{cases}$$

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Let $k \geq 2$. If $G \in \mathcal{G}_k(n)$ and $\hat{\delta}_k(G) \geq (k-1)n/k$, then $\tau_k^*(G) = n$.

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UB: $\tau_k^*(G) \leq n$.
Setting $x(v) \equiv 1/k$ gives a feasible solution to the minLP, so $\tau_k^*(G) \leq (kn) \cdot (1/k) = n$.

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Let $G = (V_1, V_2; E)$. If either V_1 or V_2 fails to have a "slack vertex" in the maxLP, then

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Each vertex in $N(v_1), N(v_2)$ has weight 1. Since $|N(v_1)|, |N(v_2)| \geq n/2$, $\tau_k^*(G) \geq n$.

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Let $G_i \leq G[N(v_i)]$, $\forall i$, so that G_i has exactly $\frac{k-1}{k} n$ vertices in each V_j.

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Each G_i satisfies the degree requirement for $\mathcal{G}_{k-1}\left(\frac{k-1}{k} n\right)$.

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Let $G_i \leq G[N(v_i)]$, $\forall i$, so that G_i has exactly $\frac{k-1}{k}n$ vertices in each V_j.

Each G_i satisfies the degree requirement for $\mathcal{G}_{k-1}\left(\frac{k-1}{k}n\right)$.

By induction,

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Linear programming

LB: $\tau_k^*(G) \geq n$. Induction Step

Let $G = (V_1, \ldots, V_k; E)$. If any V_i has no slack vertices in the maxLP, then

$$\tau_k^*(G) \geq \sum_T w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n.$$

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Future work

- Can we replace $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right) n + \epsilon n$ with $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right) n + C(H)$?

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- What probability p guarantees that, for any G with $\hat{\delta}_k(G) \geq (k-1)pn/k + \epsilon pn$, the random subgraph G_p has a K_k-tiling?

Thanks!

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My CV (with links to this and previous talks):

http://orion.math.iastate.edu/rymartin/cv/RMcv.pdf

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