An asymptotic multipartite Kühn-Osthus theorem

Ryan R. Martin\textsuperscript{1} \quad Richard Mycroft\textsuperscript{2} \quad Jozef Skokan\textsuperscript{3}

\textsuperscript{1}Iowa State University \quad \textsuperscript{2}University of Birmingham \quad \textsuperscript{3}London School of Economics

13 November 2016

AMS Eastern Sectional Meeting \#1124
Special Session on Graphs, Hypergraphs, and Set Systems
North Carolina State University, Raleigh, NC

Martin’s research partially supported by:

NSF grant DMS-0901008, NSA grant H982320-13-1-0226, Simons Foundation grant \#353292

and an Iowa State University Faculty Development Grant.
This talk is based on joint work with:

Richard Mycroft  
University of Birmingham

Jozef Skokan  
London School of Economics
Theorem (Hajnal-Szemerédi, 1970)

(Complementary form) If $G$ is a simple graph on $n$ vertices with minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{k}\right)n$$

then $G$ contains a subgraph which consists of $\lfloor n/k \rfloor$ vertex-disjoint copies of $K_k$.
The Hajnal-Szemerédi theorem

**Theorem (Hajnal-Szemerédi, 1970)**

*(Complementary form)* If $G$ is a simple graph on $n$ vertices with minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{k}\right)n$$

then $G$ contains a subgraph which consists of $\lceil n/k \rceil$ vertex-disjoint copies of $K_k$.

This is a $K_k$-tiling.
(Complementary form) If $G$ is a simple graph on $n$ vertices with minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{k}\right) n$$

then $G$ contains a subgraph which consists of $\lfloor n/k \rfloor$ vertex-disjoint copies of $K_k$.

This is a $K_k$-tiling or a $K_k$-factor or even a $K_k$-packing.
Theorem (Hajnal-Szemerédi, 1970)

\[(Complementary \ form) \text{ If } G \text{ is a simple graph on } n \text{ vertices with minimum degree}
\]
\[
\delta(G) \geq \left(1 - \frac{1}{k}\right)n
\]

then \(G\) contains a subgraph which consists of \(\lfloor n/k \rfloor\) vertex-disjoint copies of \(K_k\).

This is a \(K_k\)-tiling or a \(K_k\)-factor or even a \(K_k\)-packing. We will use “tiling” most often.
## The Hajnal-Szemerédi theorem

### Theorem (Hajnal-Szemerédi, 1970)

(Complementary form) If $G$ is a simple graph on $n$ vertices with minimum degree

$$\delta(G) \geq \left( 1 - \frac{1}{k} \right) n$$

then $G$ contains a subgraph which consists of $\lfloor n/k \rfloor$ vertex-disjoint copies of $K_k$.

This is a $K_k$-tiling or a $K_k$-factor or even a $K_k$-packing. We will use “tiling” most often.

### Notes

- $k = 2$ follows from Dirac
The Hajnal-Szemerédi theorem

Theorem (Hajnal-Szemerédi, 1970)

(Complementary form) If $G$ is a simple graph on $n$ vertices with minimum degree

$$\delta(G) \geq \left(1 - \frac{1}{k}\right)n$$

then $G$ contains a subgraph which consists of $\lfloor n/k \rfloor$ vertex-disjoint copies of $K_k$.

This is a $K_k$-tiling or a $K_k$-factor or even a $K_k$-packing. We will use “tiling” most often.

Notes

- $k = 2$ follows from Dirac
- $k = 3$ proven by Corrádi & Hajnal 1963
The Alon-Yuster theorem

**Theorem (Alon-Yuster, 1992)**

For any $\alpha > 0$ and graph $H$, there exists an $n_0 = n_0(\alpha, H)$ such that in any graph $G$ on $n \geq n_0$ vertices with

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right)n + \alpha n$$

there is an $H$-tiling of $G$ if $|V(H)|$ divides $n$. 

Komlós, Sárközy, and Szemerédi, 2001, showed that $\alpha n$ can be replaced by $C = C(H)$, but not eliminated entirely.

**Theorem (Kühn-Osthus, 2009)**

For any graph $H$, there exists an $n_0 = n_0(H)$ and a constant $C = C(H)$ such that in any graph $G$ on $n \geq n_0$ vertices with

$$\delta(G) \geq \left(1 - \frac{1}{\chi^*(H)}\right)n + C$$

there is an $H$-tiling of $G$ if $|V(H)|$ divides $n$.

This result is best possible, up to the constant $C$.

But what is $\chi^*(H)$?
The Alon-Yuster theorem

**Theorem (Alon-Yuster, 1992)**

For any $\alpha > 0$ and graph $H$, there exists an $n_0 = n_0(\alpha, H)$ such that in any graph $G$ on $n \geq n_0$ vertices with

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right)n + \alpha n$$

there is an $H$-tiling of $G$ if $|V(H)|$ divides $n$.

Komlós, Sárközy and Szemerédi, 2001, showed that $\alpha n$ can be replaced by $C = C(H)$, but not eliminated entirely.
The Alon-Yuster theorem

**Theorem (Alon-Yuster, 1992)**

For any $\alpha > 0$ and graph $H$, there exists an $n_0 = n_0(\alpha, H)$ such that in any graph $G$ on $n \geq n_0$ vertices with

$$\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n + \alpha n$$

there is an $H$-tiling of $G$ if $|V(H)|$ divides $n$.

**Theorem (Kühn-Osthus, 2009)**

For any graph $H$, there exists an $n_0 = n_0(H)$ and a constant $C = C(H)$ such that in any graph $G$ on $n \geq n_0$ vertices with

$$\delta(G) \geq \left(1 - \frac{1}{\chi^*(H)}\right) n + C$$

there is an $H$-tiling of $G$ if $|V(H)|$ divides $n$. 

This result is best possible, up to the constant $C$. 

But what is $\chi^*(H)$?
The Alon-Yuster theorem

Theorem (Kühn-Osthus, 2009)

For any graph $H$, there exists an $n_0 = n_0(H)$ and a constant $C = C(H)$ such that in any graph $G$ on $n \geq n_0$ vertices with

$$\delta(G) \geq \left(1 - \frac{1}{\chi^*(H)}\right)n + C$$

there is an $H$-tiling of $G$ if $|V(H)|$ divides $n$.

This result is best possible, up to the constant $C$.

But what is $\chi^*(H)$?
Critical chromatic number

**Definition**

Let $H$ be a graph with

- order: $h = |V(H)|$
- chromatic number: $\chi = \chi(H)$
Critical chromatic number

Definition

Let $H$ be a graph with

- order: $h = |V(H)|$
- chromatic number: $\chi = \chi(H)$
- $\sigma = \sigma(H)$ is the order of the smallest color class of $H$ among all proper $\chi$-colorings of $V(H)$. 
Critical chromatic number

**Definition**

Let $H$ be a graph with

- order: $h = |V(H)|$
- chromatic number: $\chi = \chi(H)$
- $\sigma = \sigma(H)$ is the order of the smallest color class of $H$ among all proper $\chi$-colorings of $V(H)$.

The critical chromatic number of $H$, is $\chi_{cr} = \chi_{cr}(H) = \frac{(\chi-1)h}{h-\sigma}$.
Critical chromatic number

**Definition**

Let $H$ be a graph with

- order: $h = |V(H)|$
- chromatic number: $\chi = \chi(H)$
- $\sigma = \sigma(H)$ is the order of the smallest color class of $H$ among all proper $\chi$-colorings of $V(H)$.

The critical chromatic number of $H$, is $\chi_{cr} = \chi_{cr}(H) = \frac{(\chi-1)h}{h-\sigma}$

**Fact**

For any graph $H$:

$$\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$$
Critical chromatic number

Definition

Let $H$ be a graph with

- order: $h = |V(H)|$
- chromatic number: $\chi = \chi(H)$
- $\sigma = \sigma(H)$ is the order of the smallest color class of $H$ among all proper $\chi$-colorings of $V(H)$.

The critical chromatic number of $H$, is $\chi_{cr} = \chi_{cr}(H) = \frac{(\chi-1)h}{h-\sigma}$

Fact

For any graph $H$:

$\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$

Also, $\chi_{cr}(H) = \chi(H)$ iff every proper $\chi$-coloring of $H$ is a equipartition.

\(\chi_{cr}(H)\) was defined by Komlós, 2000.
Critical chromatic number

Definition

Let $H$ be a graph with
- order: $h = |V(H)|$
- chromatic number: $\chi = \chi(H)$
- $\sigma = \sigma(H)$ is the order of the smallest color class of $H$ among all proper $\chi$-colorings of $V(H)$.

The critical chromatic number of $H$, is $\chi_{cr} = \chi_{cr}(H) = \frac{(\chi-1)h}{h-\sigma}$

\[ \chi^*(H) = \begin{cases} 
\chi_{cr}(H), & \text{if } \gcd(H) = 1; \\
\chi(H), & \text{else.}
\end{cases} \]

where $\gcd(H)$ is basically the gcd of the differences of the color classes in proper colorings of $H$. 

Definitions

Definition
The family of $k$-partite graphs with $n$ vertices in each part is denoted $G_k(n)$.

Definition
The natural bipartite subgraphs of $G$ are the ones induced by the pairs of classes of the $k$-partition.

Definition
If $G \in G_k(n)$, let $\hat{\delta}_k(G)$ denote the minimum degree among all of the natural bipartite subgraphs of $G$. 
The asymptotic Hajnal-Szemerédi theorem was solved with two different methods:

**Theorem (Keevash-Mycroft, 2013; Lo-Markström, 2013)**

Let \( k \geq 2 \) and \( \varepsilon > 0 \). There exists an \( n_0 = n_0(k, \varepsilon) \) such that if \( n \geq n_0 \), \( G \in G_k(n) \) and if

\[
\delta_k(G) \geq \left(1 - \frac{1}{k}\right)n + \varepsilon n,
\]

then \( G \) has a \( K_k \)-tiling.

Hypergraph blow-up; Absorbing method
The asymptotic Hajnal-Szemerédi theorem was solved with two different methods:

Theorem (Keevash-Mycroft, 2013; Lo-Markström, 2013)

Let \( k \geq 2 \) and \( \epsilon > 0 \). There exists an \( n_0 = n_0(k, \epsilon) \) such that if \( n \geq n_0 \), \( G \in \mathcal{G}_k(n) \) and if

\[
\hat{\delta}_k(G) \geq \left( 1 - \frac{1}{k} \right) n + \epsilon n,
\]

then \( G \) has a \( K_k \)-tiling.

Hypergraph blow-up; Absorbing method
The asymptotic Hajnal-Szemerédi theorem was solved with two different methods:

**Theorem (Keevash-Mycroft, 2013; Lo-Markström, 2013)**

Let $k \geq 2$ and $\epsilon > 0$. There exists an $n_0 = n_0(k, \epsilon)$ such that if $n \geq n_0$, $G \in G_k(n)$ and if

$$\hat{\delta}_k(G) \geq \left(1 - \frac{1}{k}\right)n + \epsilon n,$$

then $G$ has a $K_k$-tiling.

Hypergraph blow-up; Absorbing method
In a longer manuscript, Keevash and Mycroft settle the multipartite Hajnal-Szemerédi case for large $n$:


Let $k \geq 2$ and $\epsilon > 0$. There exists an $n_0 = n_0(k, \epsilon)$ such that if $n \geq n_0$, $G \in G_k(n)$ and if

$$\hat{\delta}_k(G) \geq \left(1 - \frac{1}{k}\right)n,$$

then $G$ has a $K_k$-tiling or both $k$ and $n/k$ are odd integers and $G \approx \Gamma_k(n/k)$.

The case of $k = 3$ was solved by Magyar-M. (2002). The case of $k = 4$ was solved by M.-Szemerédi (2008).

The graph $\Gamma_k(n/k)$ is one of Catlin’s “Type 2” graphs.
Catlin’s Type 2 graphs.

The red indicates non-edges between graph classes.
Theorem (Zhao, 2009)

Let $h$ be a positive integer. There exists an $n_0 = n_0(h)$ such that if $n \geq n_0$, $h | n$, and $G \in \mathcal{G}_2(n)$ with

$$\delta(G) = \hat{\delta}_2(G) \geq \begin{cases} \frac{1}{2}n + h - 1, & \text{if } n/h \text{ is odd;} \\ \frac{1}{2}n + \frac{3h}{2} - 2, & \text{if } n/h \text{ is even,} \end{cases}$$

then $G$ has a perfect $K_{h,h}$-tiling.

Moreover, there are examples that prove that this $\hat{\delta}_2$ condition cannot be improved.
Theorem (Bush-Zhao, 2012)

Let $H$ be a bipartite graph. There exists an $n_0 = n_0(H)$ and $c = c(H)$ such that if $n \geq n_0$, $|V(H)| \mid n$, and $G \in \mathcal{G}_2(n)$ with

$$\delta(G) \geq \begin{cases} 
\left(1 - \frac{1}{\chi^*(H)}\right)n + c, & \text{if } \gcd(H) = 1 \text{ or } \gcd_{cc}(H) > 1; \\
\left(1 - \frac{1}{\chi(H)}\right)n + c, & \text{if } \gcd(H) > 1 \text{ and } \gcd_{cc}(H) = 1,
\end{cases}$$

then $G$ has a perfect $H$-tiling.

The quantity $\gcd_{cc}(H)$ counts the gcd of the sizes of the connected components of $H$. 
Our results

**Theorem (M.-Skokan, 2013+)**

Let $k \geq 2$, $H$ be a graph with $\chi(H) = k$ and $\epsilon > 0$. There exists an $n_0 = n_0(H, \epsilon)$ such that if $n \geq n_0$, $G \in G_k(n)$ and if

$$\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi(H)}\right)n + \epsilon n,$$

then $G$ has an $H$-tiling.
Our results

Theorem (M.-Skokan, 2013+)

Let \( k \geq 2 \), \( H \) be a graph with \( \chi(H) = k \) and \( \epsilon > 0 \). There exists an \( n_0 = n_0(H, \epsilon) \) such that if \( n \geq n_0 \), \( G \in G_k(n) \) and if

\[
\hat{\delta}_k(G) \geq \left( 1 - \frac{1}{\chi(H)} \right) n + \epsilon n,
\]

then \( G \) has an \( H \)-tiling.

This, of course, contains the asymptotic Hajnal-Szemerédi case.
Our results

Theorem (M.-Mycroft-Skokan, 2015+)

Let $k \geq 2$, $H$ be a graph with $\chi(H) = k$, $\chi^* = \chi^*(H)$ and $\epsilon > 0$. There exists an $n_0 = n_0(H, \epsilon)$ such that if $n \geq n_0$, $G \in \mathcal{G}_k(n)$ and if

$$\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(H)}\right)n + \epsilon n,$$

then $G$ has an $H$-tiling.

The main tool is linear programming.
For any graph $G$, let $\mathcal{T}_k(G)$ denote the set of $k$-cliques of $G$. The FRACTIONAL $K_k$-TILING NUMBER, $\tau_k^*(G)$ is:

$$\tau_k^*(G) = \begin{cases} \max & \sum_{T \in \mathcal{T}_k(G)} w(T) \\ \text{s.t.} & \sum_{T \in \mathcal{T}_k(G), T \ni v} w(T) \leq 1, \quad \forall v \in V(G), \\ & w(T) \geq 0, \quad \forall T \in \mathcal{T}_k(G). \end{cases}$$
Linear programming

Definition

For any graph $G$, let $\mathcal{T}_k(G)$ denote the set of $k$-cliques of $G$. The **FRACTIONAL $K_k$-TILING NUMBER**, $\tau^*_k(G)$ is:

$$\tau^*_k(G) = \left\{ \begin{array}{l}
\max \sum_{T \in \mathcal{T}_k(G)} w(T) \\
\text{s.t.} \sum_{T \in \mathcal{T}_k(G), T \ni v} w(T) \leq 1, \ \forall v \in V(G), \\
w(T) \geq 0, \ \forall T \in \mathcal{T}_k(G).
\end{array} \right.$$  

Theorem

Let $k \geq 2$. If $G \in \mathcal{G}_k(n)$ and $\delta_k(G) \geq (k - 1)n/k$, then $\tau^*_k(G) = n$. 
Complementary Slackness Theorem of LPs.

Let $\tau_k^*(G) = \max \left\{ \sum_{T \in \mathcal{T}_k(G)} w(T) \mid \sum_{T \in \mathcal{T}_k(G), T \ni v} w(T) \leq 1, \forall v \in V(G), w(T) \geq 0, \forall T \in \mathcal{T}_k(G) \right\}$.

Theorem

Let $k \geq 2$. If $G \in \mathcal{G}_k(n)$ and $\hat{\delta}_k(G) \geq (k - 1)n/k$, then $\tau_k^*(G) = n$.

The proof is by induction on $k$ and uses both the Duality Theorem and Complementary Slacksness Theorem of LPs.
Theorem

Let \( k \geq 2 \). If \( G \in \mathcal{G}_k(n) \) and \( \delta_k(G) \geq (k - 1)n/k \), then \( \tau_k^*(G) = n \).

The proof is by induction on \( k \) and uses both the Duality Theorem and Complementary Slackness Theorem of LPs.

Duality Theorem:

\[
\tau_k^*(G) = \left\{ \begin{array}{l}
\max \sum w(T) \\
\text{s.t.} \sum_{T \ni v} w(T) \leq 1, \forall v, \\
w(T) \geq 0, \forall T.
\end{array} \right. = \left\{ \begin{array}{l}
\min \sum x(v) \\
\text{s.t.} \sum_{v \in T} x(v) \geq 1, \forall T, \\
x(v) \geq 0, \forall v.
\end{array} \right.
\]
Theorem

Let $k \geq 2$. If $G \in \mathcal{G}_k(n)$ and $\hat{\delta}_k(G) \geq (k - 1)n/k$, then $\tau_k^*(G) = n$.

The proof is by induction on $k$ and uses both the Duality Theorem and Complementary Slackness Theorem of LPs.

Duality Theorem:

$$\tau_k^*(G) = \begin{cases} \max & \sum w(T) \\ s.t. & \sum_{T \ni v} w(T) \leq 1, \quad \forall v, \quad \sum_{T \ni v} w(T) \geq 0, \quad \forall T. \end{cases}$$

$$= \begin{cases} \min & \sum x(v) \\ s.t. & \sum_{v \in T} x(v) \geq 1, \quad \forall T, \quad x(v) \geq 0, \quad \forall v. \end{cases}$$

UB: $\tau_k^*(G) \leq n$.

Setting $x(v) = 1/k$ gives a feasible solution to the minLP, so $\tau_k^*(G) \leq (kn) \cdot (1/k) = n$. 

An asymptotic multipartite Kühn-Osthus theorem...
Linear programming

Theorem

Let $k \geq 2$. If $G \in G_k(n)$ and $\hat{\delta}_k(G) \geq (k-1)n/k$, then $\tau_k^*(G) = n$.

$$
\tau_k^*(G) = \begin{cases} 
\max & \sum w(T) \\
\text{s.t.} & \sum_{T \ni v} w(T) \leq 1, \quad \forall v, \\
& w(T) \geq 0, \quad \forall T.
\end{cases}
= \begin{cases} 
\min & \sum x(v) \\
\text{s.t.} & \sum_{v \in T} x(v) \geq 1, \quad \forall T, \\
x(v) \geq 0, \quad \forall v.
\end{cases}
$$

UB: $\tau_k^*(G) \leq n$.

Setting $x(v) \equiv 1/k$ gives a feasible solution to the minLP, so $\tau_k^*(G) \leq (kn) \cdot (1/k) = n$.

LB: $\tau_k^*(G) \geq n$. Base Case: $k = 2$. 

Martin (Iowa State University University of Birmingham London School of Economics)
Linear programming

\[ \tau_k^*(G) = \begin{cases} 
\max & \sum w(T) \\
\text{s.t.} & \sum_{T \ni v} w(T) \leq 1, \quad \forall v, \quad \sum w(T) \geq 0, \quad \forall T. 
\end{cases} \]

\[ = \begin{cases} 
\min & \sum x(v) \\
\text{s.t.} & \sum_{v \in T} x(v) \geq 1, \quad \forall T, \\
& x(v) \geq 0, \quad \forall v. 
\end{cases} \]

**UB:** \( \tau_k^*(G) \leq n. \)

Setting \( x(v) \equiv 1/k \) gives a feasible solution to the minLP, so

\[ \tau_k^*(G) \leq (kn) \cdot (1/k) = n. \]

**LB:** \( \tau_k^*(G) \geq n. \) Base Case: \( k = 2. \)

Let \( G = (V_1, V_2; E) \). If either \( V_1 \) or \( V_2 \) fails to have a "slack vertex" in the maxLP, then

\[ \tau_k^*(G) \geq \sum_T w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n. \]
Linear programming

\[ \tau_k^*(G) = \begin{cases} \max_{s.t.} & \sum_{T \ni v} w(T) \leq 1, \forall v, \\ & \sum_{T \ni v} w(T) \geq 0, \forall T. \end{cases} \]

\[ = \begin{cases} \min_{s.t.} & \sum_{v \in T} x(v) \geq 1, \forall T, \\ & \sum_{v \in V} x(v) \geq 0, \forall v. \end{cases} \]

**LB:** \( \tau_k^*(G) \geq n. \) Base Case: \( k = 2. \)

Let \( G = (V_1, V_2; E). \) If either \( V_1 \) or \( V_2 \) fails to have a “slack vertex” in the maxLP, then

\[ \tau_k^*(G) \geq \sum_{T} w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n. \]

If \( v_1 \in V_1 \) and \( v_2 \in V_2 \) are slack, then we may assume \( x(v_1) = x(v_2) = 0 \) (Complementary Slackness).
Linear programming

\[ \tau_k^*(G) = \begin{cases} 
\max & \sum T \in V \, w(T) \\
\text{s.t.} & \sum_{T \ni v} w(T) \leq 1, \quad \forall v, \\
& w(T) \geq 0, \quad \forall T. 
\end{cases} \]

\[ = \begin{cases} 
\min & \sum_{v \in V} x(v) \\
\text{s.t.} & \sum_{v \in T} x(v) \geq 1, \quad \forall T, \\
& x(v) \geq 0, \quad \forall v. 
\end{cases} \]

LB: \( \tau_k^*(G) \geq n. \) Base Case: \( k = 2. \)

Let \( G = (V_1, V_2; E). \) If either \( V_1 \) or \( V_2 \) fails to have a “slack vertex” in the maxLP, then

\[ \tau_k^*(G) \geq \sum_T w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n. \]

If \( v_1 \in V_1 \) and \( v_2 \in V_2 \) are slack, then we may assume \( x(v_1) = x(v_2) = 0 \) (Complementary Slackness).

Each vertex in \( N(v_1), N(v_2) \) has weight 1. Since \( |N(v_1)|, |N(v_2)| \geq n/2, \)

\[ \tau_k^*(G) \geq n. \]
**Linear programming**

\[ \tau_k^*(G) = \begin{cases} 
\max & \sum w(T) \\
\text{s.t.} & \sum_{T \ni v} w(T) \leq 1, \quad \forall v, \\
& w(T) \geq 0, \quad \forall T. 
\end{cases} = \begin{cases} 
\min & \sum x(v) \\
\text{s.t.} & \sum_{v \in T} x(v) \geq 1, \quad \forall T, \\
& x(v) \geq 0, \quad \forall v. 
\end{cases} \]

**LB:** \( \tau_k^*(G) \geq n. \) Induction Step

Let \( G = (V_1, \ldots, V_k; E). \) If any \( V_i \) has no slack vertices in the maxLP, then

\[ \tau_k^*(G) \geq \sum_{T} w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n. \]
Linear programming

\[ \tau^*_k(G) = \begin{cases} \max \sum w(T) & \text{s.t.} \quad \sum_{T \ni v} w(T) \leq 1, \quad \forall v, \\ w(T) \geq 0, \quad \forall T. \end{cases} = \begin{cases} \min \sum x(v) & \text{s.t.} \quad \sum_{v \in T} x(v) \geq 1, \quad \forall T, \\ x(v) \geq 0, \quad \forall v. \end{cases} \]

LB: \( \tau^*_k(G) \geq n \). Induction Step

Let \( G = (V_1, \ldots, V_k; E) \). If any \( V_i \) has no slack vertices in the maxLP, then

\[ \tau^*_k(G) \geq \sum_{T} w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n. \]

If \( v_i \in V_i, \forall i \), are slack, then we may assume \( x(v_i) = 0, \forall i \).
Linear programming

\[ \tau_k^*(G) = \begin{cases} \max & \sum w(T) \\ \text{s.t.} & \sum_{T \ni v} w(T) \leq 1, \quad \forall v, \\ & w(T) \geq 0, \quad \forall T. \end{cases} = \begin{cases} \min & \sum x(v) \\ \text{s.t.} & \sum_{v \in T} x(v) \geq 1, \quad \forall T, \\ & x(v) \geq 0, \quad \forall v. \end{cases} \]

**LB:** \( \tau_k^*(G) \geq n. \) Induction Step

Let \( G = (V_1, \ldots, V_k; E) \). If any \( V_i \) has no slack vertices in the maxLP, then

\[ \tau_k^*(G) \geq \sum_{T} w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n. \]

If \( v_i \in V_i, \forall i, \) are slack, then we may assume \( x(v_i) = 0, \forall i. \)

Let \( G_i \leq G[N(v_i)], \forall i, \) so that \( G_i \) has exactly \( \frac{k-1}{k} n \) vertices in each \( V_j. \)
**LB:** \( \tau^*_k(G) \geq n. \) Induction Step

Let \( G = (V_1, \ldots, V_k; E) \). If any \( V_i \) has no slack vertices in the maxLP, then

\[
\tau^*_k(G) \geq \sum_{T} w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n.
\]

If \( v_i \in V_i, \forall i \), are slack, then we may assume \( x(v_i) = 0, \forall i \).

Let \( G_i \leq G[N(v_i)], \forall i \), so that \( G_i \) has exactly \( \frac{k-1}{k} n \) vertices in each \( V_j \).

Each \( G_i \) satisfies the degree requirement for \( G_{k-1}(\frac{k-1}{k} n) \).
Linear programming

**LB:** $\tau_k^*(G) \geq n$. Induction Step

Let $G = (V_1, \ldots, V_k; E)$. If any $V_i$ has no slack vertices in the maxLP, then

$$\tau_k^*(G) \geq \sum_{T} w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n.$$

If $v_i \in V_i$, $\forall i$, are slack, then we may assume $x(v_i) = 0$, $\forall i$.

Let $G_i \leq G[N(v_i)]$, $\forall i$, so that $G_i$ has exactly $\frac{k-1}{k} n$ vertices in each $V_j$.

Each $G_i$ satisfies the degree requirement for $G_{k-1}\left(\frac{k-1}{k} n\right)$.

By induction,

$$(k - 1)\tau_k^*(G) \geq \sum_{i=1}^{k} \sum_{v \in V(G_i)} x(v) \geq \sum_{i=1}^{k} \frac{k - 1}{k} n = (k - 1)n.$$
**LB:** $\tau_k^*(G) \geq n$. Induction Step

Let $G = (V_1, \ldots, V_k; E)$. If any $V_i$ has no slack vertices in the maxLP, then

$$\tau_k^*(G) \geq \sum_{T} w(T) = \sum_{v \in V_i} \sum_{T \ni v} w(T) = \sum_{v \in V_i} 1 = n.$$

If $v_i \in V_i$, $\forall i$, are slack, then we may assume $x(v_i) = 0$, $\forall i$.

Let $G_i \leq G[N(v_i)]$, $\forall i$, so that $G_i$ has exactly $\frac{k-1}{k}n$ vertices in each $V_j$.

Each $G_i$ satisfies the degree requirement for $G_{k-1}\left(\frac{k-1}{k}n\right)$.

By induction,

$$(k - 1)\tau_k^*(G) \geq \sum_{i=1}^{k} \sum_{v \in V(G_i)} x(v) \geq \sum_{i=1}^{k} \frac{k-1}{k}n = (k - 1)n. \quad \square$$
Can we replace $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right) n + \epsilon n$ with $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right) n + C(H)$?
Future work

- Can we replace $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right) n + \epsilon n$ with $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right) n + C(H)$?

- Is $\hat{\delta}_k(G) \geq (k - 1)n/k + \epsilon n$ sufficient to force the $k^{th}$ power of a Hamilton cycle?
  (Related to Bollobás-Komlós conjecture on bandwidth)
Future work

- Can we replace $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right)n + \epsilon n$ with $\hat{\delta}_k(G) \geq \left(1 - \frac{1}{\chi^*(G)}\right)n + C(H)$?

- Is $\hat{\delta}_k(G) \geq (k - 1)n/k + \epsilon n$ sufficient to force the $k^{th}$ power of a Hamilton cycle? (Related to Bollobás-Komlós conjecture on bandwidth)

- What probability $p$ guarantees that, for any $G$ with $\hat{\delta}_k(G) \geq (k - 1)n/k + \epsilon n$, the random subgraph $G_p$ has a $K_k$-tiling?
My home page:

http://orion.math.iastate.edu/rymartin

My CV (with links to this and previous talks):

http://orion.math.iastate.edu/rymartin/cv/RMcv.pdf

Contact me:

rymartin@iastate.edu