# Planar Turán number of the 6-Cycle

Ryan Martin

Iowa State University

Joint work with:
Debarun Ghosh, Ervin Győri, Addisu Paulos, Chuanqi Xiao

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# Mantel's Theorem

## Definition

*The maximum number of edges in n-vertex, H-free graph is*

$$\mathrm{ex}(n, H).$$

## Fact

*An n-vertex graph has* $\leq \binom{n}{2}$ *edges.*

## Theorem (Mantel, 1907)

*An n-vertex* $K_3$*-free graph has* $\leq \left\lfloor \frac{n^2}{4} \right\rfloor$ *edges.*

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*The maximum number of edges in n-vertex, H-free planar graph is*

$$\mathrm{ex}_{\mathcal{P}}(n, H).$$

## Theorem (via Euler, 1758)

*An n-vertex planar graph has* $\leq 3n - 6$ *edges or it is* $K_2$.

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*Proof.* If $G$ is not a tree, then

$$2e = \sum_{\phi \in F(G)} \deg(\phi) \geq f \cdot 4 = (e - n + 2) \cdot 4$$

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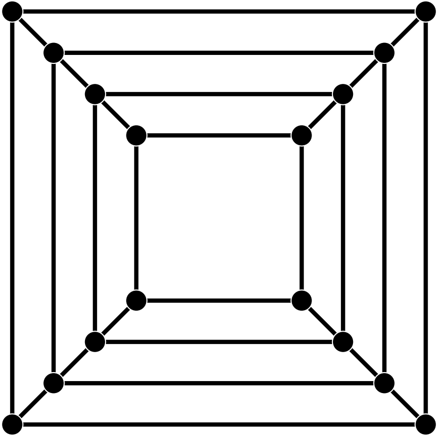
$$\mathrm{ex}_{\mathcal{P}}(n, K_3) = 2(n-2), \qquad \textit{for all } n \geq 3.$$



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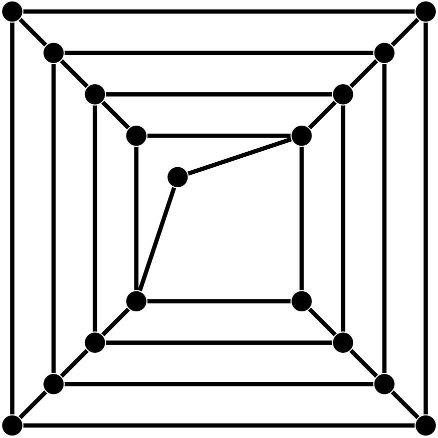

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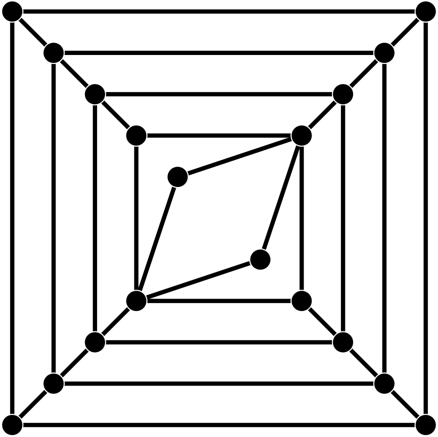

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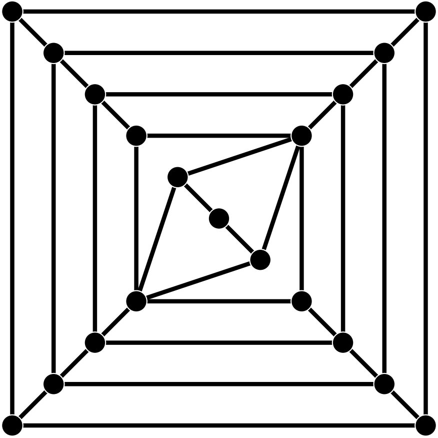

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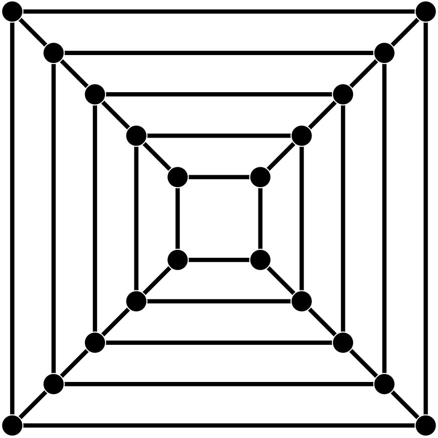

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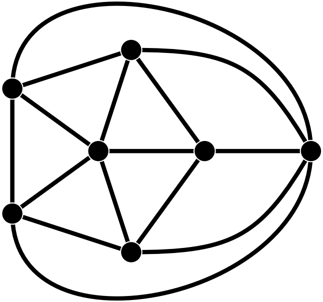

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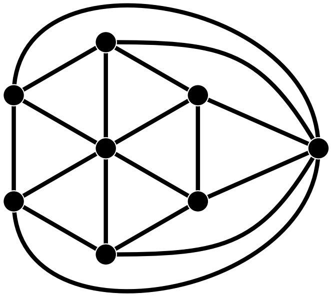

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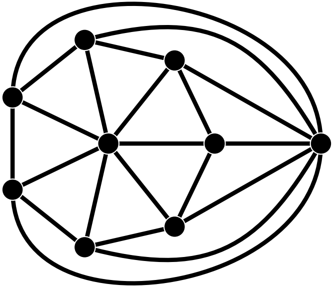

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$$\mathrm{ex}_{\mathcal{P}}(n, K_5) = 3(n-2), \qquad \textit{for all } n \geq 3.$$

# Cycle results

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$$\mathrm{ex}_{\mathcal{P}}(n, C_3) = 2(n-2) \qquad \textit{for all } n \geq 3.$$

## Theorem (Dowden, 2016)

$$\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq \frac{15}{7}(n-2), \qquad \textit{for all } n \geq 4.$$

*Equality holds for all* $n \cong 30 \pmod{70}$.

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$$\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5}{2}n - 7, \qquad \textit{for all } n \geq 18.$$

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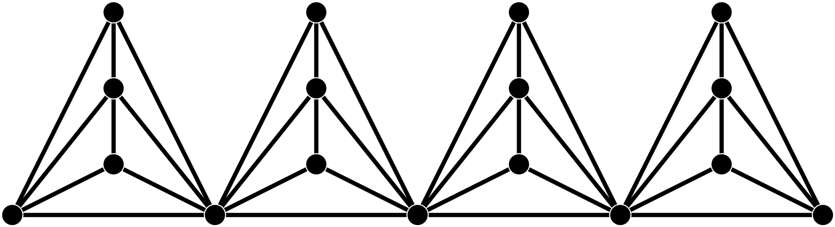
The proof:

1. Why is $n \geq 18$ necessary?
2. Which construction gives $e(G) = \frac{5}{2}n - 7$ if $n \cong 10 \pmod{18}$?
3. How do we establish that planar, $C_6$-free implies $e(G) \leq \frac{5}{2}n - 7$?

# (i) Construction for $n = 17$



$$|V(G)| = 17$$

$$|E(G)| = 36 > 35.5 = \frac{5}{2} \cdot 17 - 7$$

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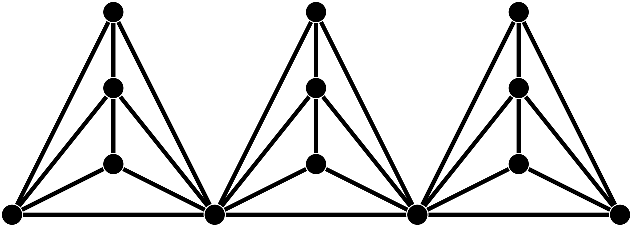

$$|V(G)| = 13$$

$$|E(G)| = 27 > 25.5 = \frac{5}{2} \cdot 16 - 7$$

# (ii) Constructions for $n \cong 10 \pmod{18}$: Underlying graph

**Theorem**

*A n-vertex ($n \geq 4$) girth-g planar graph has* $\leq \max\left\{\frac{g}{g-2}(n-2), n-1\right\}$ *edges.*

**Lemma**

*For every $k \geq 0$, there is a girth-7 graph $G_0^k$ such that*

$$n = 10k + 7 \qquad n_2 = 2k + 7$$

$$e = 14k + 7 = \frac{7}{5}(n-2) \qquad n_3 = 8k.$$

*For every $k \geq 1$, there is a girth-7 graph $H_0^k$ such that*

$$n = 10k + 2 \qquad n_2 = 2k + 6$$

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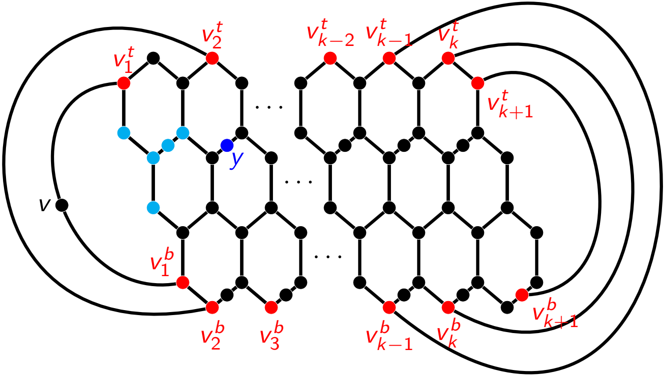


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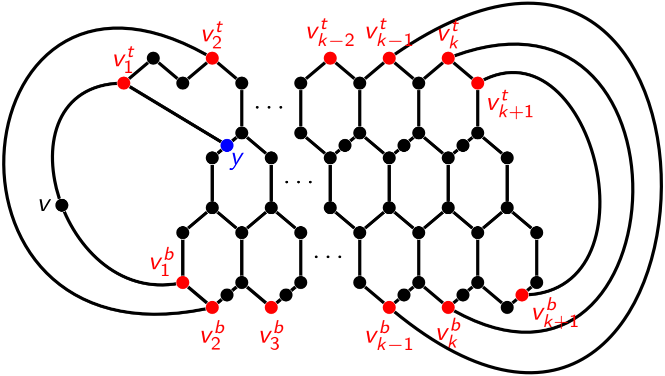


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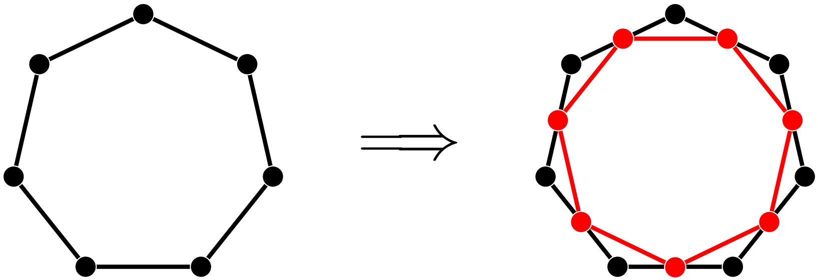
# (ii) Constructions for $n \cong 10 \pmod{18}$: Construction

**Given**

A girth-7 graph, $G_0$ with vertices of degree 2 and 3.

(1) For every edge, add a "halving vertex":



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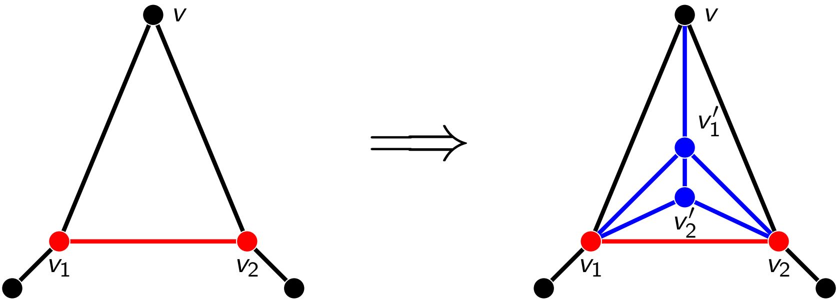

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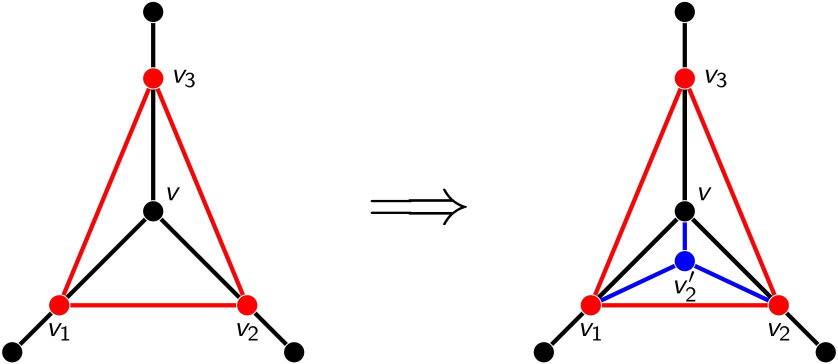
(3) For every degree-3 vertex, $v$, replace it with a $K_5^-$ as follows:



# (ii) Constructions for $n \cong 10 \pmod{18}$: Construction

**Given**

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Let $G_0$ be chosen such that

$$\begin{aligned} v(G_0) & & = v \\ e(G_0) &= \frac{7}{5}(v(G_0) - 2) & = \frac{7v - 14}{5} \end{aligned}$$

The resulting graph $G$ has the following:

$$\begin{aligned} v(G) &= v(G_0) + e(G_0) + 2n_2(G_0) + n_3(G_0) & \\ &= v + \frac{7v - 14}{5} + 2\left(\frac{v + 28}{5}\right) + \frac{4v - 28}{5} & = \frac{18v + 14}{5} \\ e(G) &= 9v(G_0) & = 9v. \end{aligned}$$

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Hence,

$$e(G) = 9\left(\frac{5v(G)-14}{18}\right) = \frac{5}{2}v(G) - 7.$$

Moreover, if $v \cong 2 \pmod{5}$, then $v(G) \cong 10 \pmod{18}$.

# (iii) Proof for upper bound: Triangular-blocks

**Theorem (Ghosh-Győri-M-Paulos-Xiao, 2020+)**

$$\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5}{2}n - 7, \qquad \textit{for all } n \geq 18.$$

We may assume: • No vertex degree 2. • No cut-vertex.

KEY IDEA: Partition the edges into *triangular-blocks*:

**Definition**

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## For each triangular-block $B \in \mathcal{B}$:

- $e(B)$ is the number of edges in $B$.
- $n(B)$ is the number of vertices in $B$.
  If a vertex is in $d$ blocks, it assigns $1/d$ to each block.
- $f(B)$ is the number of faces in $B$.
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  ...unless it has two edges of a $K_5^-$, then more complicated.

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If $G$ has $n$ vertices, $e$ edges and $f$ faces, then

$$2e - 5n + 14 = 7f + 2n - 5e = \sum_{B \in \mathcal{B}} \left(7f(B) + 2n(B) - 5e(B)\right).$$

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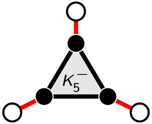
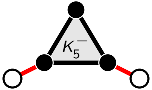
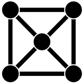
| $B$ | Diagram | $f(B) \leq$ | $n(B) \leq$ | $e(B) =$ | $7f + 2n - 5e \leq$ |
|---|---|---|---|---|---|
| $B_{5,a}$ |  | $5 + \frac{3}{7}$ | $2 + \frac{3}{2}$ | 9 | 0 |
| $B_{5,a}$ |  | $5 + \frac{2}{7}$ | $3 + \frac{2}{2}$ | 9 | 0 |
| $B_{5,b}$ |  | $4 + \frac{4}{7}$ | $3 + \frac{2}{2}$ | 8 | 0 |

Red edges do not belong to the block but indicate incidence.

# (iii) Proof for upper bound: Triangular-blocks

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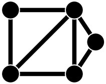
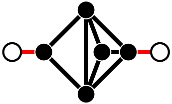
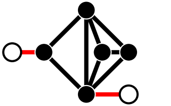
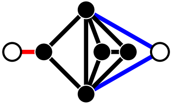
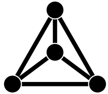
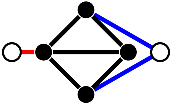
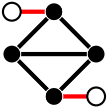
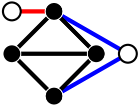
| $B_{5,c}$ |  | $3 + \frac{5}{7}$ | $3 + \frac{2}{2}$ | 7 | $-1$ |
| $B_{5,d}$ |  | $4 + \frac{4}{7}$ | $3 + \frac{2}{2}$ | 8 | 0 |
| $B_{5,d}$ |  | $4 + \frac{4}{7}$ | $3 + \frac{2}{2}$ | 8 | 0 |
| $B_{5,d}$ |  | $4 + \frac{2}{4} + \frac{2}{7}$ | $2 + \frac{3}{2}$ | 8 | $\frac{1}{2}$ ⋆ |

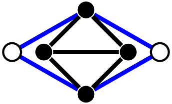
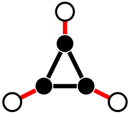
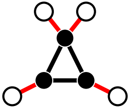
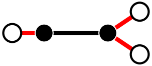
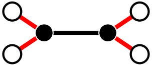
Red edges do not belong to the block but indicate incidence.
Blue edges do not belong to the block and indicate an adjacent 4-face.

# (iii) Proof for upper bound: Triangular-blocks

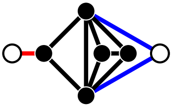
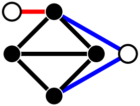
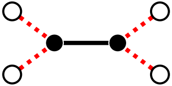
| $B$ | Diagram | $f(B) \leq$ | $n(B) \leq$ | $e(B) =$ | $7f + 2n - 5e \leq$ |
|---|---|---|---|---|---|
| $B_{4,a}$ |  | $3 + \frac{3}{7}$ | $2 + \frac{2}{2}$ | 6 | 0 |
| $B_{4,b}$ |  | $2 + \frac{2}{4} + \frac{2}{7}$ | $1 + \frac{3}{2}$ | 5 | $-\frac{1}{2}$ |
| $B_{4,b}$ |  | $2 + \frac{4}{7}$ | $2 + \frac{2}{2}$ | 5 | $-1$ |
| $B_{4,b}$ |  | $2 + \frac{2}{4} + \frac{2}{7}$ | $2 + \frac{1}{3} + \frac{1}{2}$ | 5 | $\frac{1}{6}$ ⋆ |

Red edges do not belong to the block but indicate incidence.
Blue edges do not belong to the block and indicate an adjacent 4-face.

# (iii) Proof for upper bound: Triangular-blocks

| $B$ | Diagram | $f(B) \leq$ | $n(B) \leq$ | $e(B) =$ | $7f + 2n - 5e \leq$ |
|---|---|---|---|---|---|
| $B_{4,b}$ |  | $2 + \frac{2}{4} + \frac{2}{4}$ | $2 + \frac{2}{3}$ | 5 | $\frac{4}{3}$ $\star$ |
| $B_3$ |  | $1 + \frac{2}{7} + \frac{1}{4}$ | $\frac{3}{2}$ | 3 | $-\frac{5}{4}$ |
| $B_3$ |  | $1 + \frac{3}{4}$ | $\frac{2}{2} + \frac{1}{3}$ | 3 | $-\frac{1}{12}$ |
| $B_2$ |  | $\frac{1}{4} + \frac{1}{7}$ | $\frac{2}{2}$ | 1 | $-\frac{1}{4}$ |
| $B_2$ |  | $\frac{1}{4} + \frac{1}{7}$ | $\frac{1}{2} + \frac{1}{3}$ | 1 | $-\frac{7}{12}$ |
| $B_2$ |  | $\frac{1}{4} + \frac{1}{5}$ | $\frac{2}{3}$ | 1 | $-\frac{31}{60}$ |

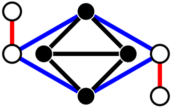
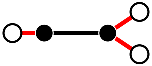
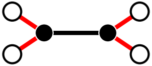
# (iii) Proof for upper bound: Triangular-blocks

| $B$ | Diagram | $f(B) \leq$ | $n(B) \leq$ | $e(B) =$ | $7f + 2n - 5e \leq$ |
|---|---|---|---|---|---|
| $B_{5,d}$ |  | $4 + \frac{2}{4} + \frac{2}{7}$ | $2 + \frac{3}{2}$ | 8 | $\frac{1}{2}$ ⋆ |
| $B_{4,b}$ |  | $2 + \frac{2}{4} + \frac{2}{7}$ | $2 + \frac{1}{3} + \frac{1}{2}$ | 5 | $\frac{1}{6}$ ⋆ |
| $B_2$ |  | | | | $-\frac{1}{4}$ |

## Blue edges:

- Form a $K_2$ triangular-block.
- Cannot be incident to two 4-faces.

# (iii) Proof for upper bound: Triangular-blocks

| $B$ | Diagram | $f(B) \leq$ | $n(B) \leq$ | $e(B) =$ | $7f + 2n - 5e \leq$ |
|---|---|---|---|---|---|
| $B_{4,b}$ |  | $2 + \frac{2}{4} + \frac{2}{4}$ | $2 + \frac{2}{3}$ | $5$ | $\frac{4}{3}$ ⋆ |
| $B_2$ |  | $\frac{1}{4} + \frac{1}{7}$ | $\frac{1}{2} + \frac{1}{3}$ | $1$ | $-\frac{7}{12}$ |
| $B_2$ |  | $\frac{1}{4} + \frac{1}{5}$ | $\frac{2}{3}$ | $1$ | $-\frac{31}{60}$ |

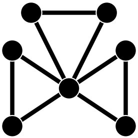
## Blue edges:

- Form a $K_2$ triangular-block.
- Cannot be incident to two 4-faces.

# Open problems

- Obtain bounds on $\mathrm{ex}_{\mathcal{P}}(n, C_k)$ for any $k \geq 7$.
- Obtain bounds for $\mathrm{ex}_{\mathcal{P}}(n, \{C_k, C_\ell\})$ for distinct $k, \ell \geq 3$
- [Lan-Shi-Song] If $n \geq 15$, then

$$\left\lfloor \frac{5n}{2} \right\rfloor \leq \mathrm{ex}_{\mathcal{P}}(n, K_1 \vee 2K_3) \leq \frac{17}{6}n - 4$$



$K_1 \vee 3K_2$

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