On diamond-free subposets of the Boolean lattice: An application of flag algebras

by

Lucas Kramer

A thesis submitted to the graduate faculty

in partial fulfillment of the requirements for the degree of

Doctorate of Philosophy

Major: Mathematics

Program of Study Committee:

Ryan Martin, Major Professor

Leslie Hogben, Stephen Willson, Roger Maddux, Zhijun Wu, Michael Young

Iowa State University

Ames, Iowa

2014

DEDICATION

I would like to dedicate this thesis to my wife Kelly. Without her support and understanding I would not have been able to complete this work.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

ACKNOWLEDGEMENTS

I wish to thank Ryan Martin, whose guidance and suggestions were necessary to complete this project and dissertation.

I also wish to thank Jakub Sliacan whose London School of Economics Master's dissertation gave me the inspiration for my approach in Chapter 2. He is currently studying for his Ph.D. at the University of Warwick.

ABSTRACT

The Boolean lattice of dimension two, also known as the diamond, consists of four distinct elements with the following property: $A \subset B, C \subset D$. The 3-crown consists of six distinct elements with the following property: $A, B \subset D$ and $B, C \subset E$ and $A, C \subset F$. A $\mathcal{P}$-free family in the n-dimensional Boolean lattice is a subposet such that no collection of elements form the poset $\mathcal{P}$. Note that the posets are not induced and may contain additional relations.

There is a diamond-free family in the n-dimensional Boolean lattice of size $(2-o(1))\binom{n}{\lfloor n/2 \rfloor}$. In this dissertation, we prove that any diamond-free family in the n-dimensional Boolean lattice has size at most $(2.25+o(1))\binom{n}{\lfloor n/2 \rfloor}$. Furthermore, we show that the so-called Lubell function of a diamond-free family in the n-dimensional Boolean lattice which contains the empty set is at most $2.25+o(1)$, which is asymptotically best possible.

There is a 3-crown-free family in the n-dimensional Boolean lattice of size $\binom{n}{\lfloor n/2 \rfloor}$. In this dissertation, we prove that any 3-crown-free family in the n-dimensional Boolean lattice has size at most $(2\sqrt{3}-2)\binom{n}{\lfloor n/2 \rfloor}$.

CHAPTER 1. Introduction

We begin by giving some basic background in graph theory, extremal graph theory, and poset theory. There are far too many areas within each of these to include but we will include many of the basic concepts the our results are based on. These will all be assumed to be familiar to the reader in later sections. If the reader is familiar with basic graph theory, one may choose to begin with subsection 1.1.1. If the reader is familiar with extremal graph theory, one may choose to begin with section 1.2.

1.1 Graph Theory

In this section we offer some basic definitions and and concepts within graph theory. We will eventually define many objects, and we will defer defining these objects until needed in the text. We will only define objects and concepts that will be used later in this dissertation. If the reader wishes for more general graph theory references please consult Diestel [Die05] and Bollobás [Bol98].

We adopt standard notation for common sets: $\mathbb{R}$ and $\mathbb{N}$ denote the real and natural numbers respectively. Set-theoretic difference and symmetric difference of A, B will be denoted by $A - B := \{x : (x \in A) \wedge (x \notin B)\}$ where $\wedge$ is a logical *and* and $A \triangle B = \{x : (x \in A - B) \vee (x \in B - A)\}$ where $\vee$ is a logical *or* respectively. The ceiling and floor functions will be denote by $\lceil x \rceil$ and $\lfloor x \rfloor$ respectively. Further notation will be introduced when needed.

A (simple) graph G is a pair $(V(G), E(G))$, with $V(G)$ a finite set and $E(G)$ a set of 2-element subsets, $\{x, y\}$ of $V(G)$. In this dissertation we will not consider "loops" where edges are allowed to be multi-sets or multi-graphs where we allow multiple copies of the same edge. Elements of $V(G)$ are called *vertices* of G and elements of $E(G)$ are called *edges* of G. If $E(G)$

is a family of r-element subsets of $V(G)$ with $r > 2$, we call G an *r-regular hypergraph* or *r-graph*. Unless otherwise noted we will assume $r = 2$. Let the *order* of a graph G, $|V(G)|$, be the cardinality of the vertex set of G and the *size* of G, $|E(G)|$, to be the cardinality of the edge set of G. If $x, y \in V(G)$ where x and y are connected by an edge, we write $xy \in E(G)$ and say x and y are *adjacent* in G. An edge $xy \in E(G)$ is said to be incident with its *endvertices* x and y. Vertices adjacent to x are collectively called the *(open) neighborhood* of x and will be denoted by $N(x)$. We define the *closed neighborhood* of $N[x] := N(x) \cup \{x\}$. The *degree* $\deg(x)$ of $x \in V(G)$ is $\deg(G) := |N(x)|$. I.e. $\deg(x)$ is the number of edges incident to x, which is the same as the number of edges adjacent to x. Please note that in all of these definitions we eliminate the reference to G whenever it is clear from context.

Graphs have many different representations (see Example 1). Different representations are advantageous in different settings. Example 1.1 and 1.2 are useful when dealing with computers. Example 1.4 is useful when placing the graph in text. Example 1.3 can be useful when visualizing graphs, but we must be careful with this representation as as both graphs in Example 1.4 are the same graph. This becomes clear after we introduce graph isomorphisms.

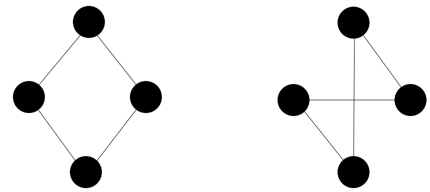
Example 1 *Different representations of the same graph G*

1. *By definition $G = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\})$*

2. *By adjacency matrix*

$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

3. By drawing



(a) G (b) G

4. By name $G = C_4$

If G and G' are different representations of the same graph, we say that they are *isomorphic* to each other. Next, we give the official definition of graph isomorphism.

Definition 2 (Graph Isomorphism) *Two graphs G and G' are isomorphic, denoted by $G \cong G'$, if there exists a bijective function $\eta : V(G) \to V(G')$ such that $xy \in E(G)$ if and only if $\eta(x)\eta(y) \in E(G')$.*

We may think of this definition as two graphs are isomorphic if there is a bijective map on the vertices that preserves adjacency. We stated before that all the examples in example 1 represent the same graph. It is more accurate but still abusive in notation to saw that all the examples in example 1 represent graphs that are isomorphic. This is an important concept and will be vital to our development of flag algebras.

Next we introduce the concept of subgraph.

Definition 3 (Subgraph) *Let G be a graph. A graph G' with $V(G') \subseteq V(G)$ and $E(G') \subseteq \{xy \in E(G) : x, y \in V(G')\}$ is a subgraph of G, denoted by $G' \subseteq G$. If, in addition, $E(G') = \{xy \in E(G) : x, y \in V(G')\}$ then G' is an induced subgraph of G, this is also denoted by $G[V(G')]$.*

There are several graphs that are well known and appear often in all parts of graph theory. We have special names for each of them such at C_3 is the triangle and C_4 is the 4-cycle. We now introduce select graphs that will appear later.

A *path*, denoted P_n, is a graph with vertex set $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1} : 1 \leq i < n\}$. Note that if we were to travel along edges from v_i to v_j we would have to travel through vertices $v_{i+1}, \ldots, v_{j-1}$. The *length* of a path is the number of edges in it. A (u, v)-path is a path starting at u and ending at v in G. Vertices u and v in a graph G are said to be *connected* if there exists a (u, v)-path. A graph G is said to be *connected* if every pair of vertices in G is connected. The *distance* of two vertices u, v in G is the minimum length of a (u, v)-path, denoted $d(u, v)$. A *component* of G is the graph induced on a maximal set of connected vertices. If u, v are in different components of G, then their distance is defined to be ∞.

A *cycle* C_n on n vertices is a closed path, i.e. $V(C_n) = V(P_n)$ and $E(C_n) = E(P_n) \cup \{v_1 v_n\}$. C_n is called an *n-cycle*. The *length* of a cycle, $l(C_n)$ is defined to be its size, $l(C_n) = n$. A graph G is *acyclic* if it does not contain any cycle as a subgraph.

A *bipartite graph* is a graph G where $V(G)$ is partitioned into $V'(G)$ and $V''(G)$ and $E(G) \subseteq \{xy : x \in V'(G), y \in V''(G)\}$. I.e. a bipartite graph is a graph G where the vertex set is partitioned into two parts and edges only exist between the partitions. If $E(G) = \{xy : x \in V'(G), y \in V''(G)\}$ we call G the *complete bipartite graph*, denoted $K_{i,j}$ where $i = |V'(G)|$ and $j = |V''(G)|$. Note that bipartite is equivalent to saying a graph G contains no odd length cycles. The proof of this can be found in Diestel [Die05] and is omitted. An *r-partite graph* is a graph G where $V(G)$ is partitioned into r pieces and edges only exist between the partitions.

A graph G on n vertices is said to be *complete* if $E(G)$ contains every distinct pair of vertices; G is then denoted as K_n. Note that $|E(G)| = \binom{n}{2}$. The graph K_n is also called a *n-clique*. This notation more often appears when K_n is a subgraph of a larger graph G'. A graph G is the *empty* graph if $E(G) = \emptyset$. An empty graph E on n vertices is sometimes called an *independent set of size* n but is usually within the context of when E is an induced subgraph of a larger graph G', i.e. $E = G'[V(E)]$.

Let G be any graph on n vertices. Then $\overline{G}$ denotes the *complement* of G and is defined to be a graph with vertices $V(\overline{G}) = V(G)$ and $E(\overline{G}) = E(K_n) - E(G)$. Note that a complete graph and empty graph on n vertices are complements. Also the graph

$$G = (\{1,2,3,4\}, \{\{1,2\},\{2,3\},\{3,4\}\})$$

is isomorphic to its complement

$$\overline{G} = (\{1,2,3,4\}, \{\{1,3\},\{1,4\},\{2,4\}\}).$$

One graph concept that is very closely related to our result is the concept of *Turán graphs* $T(n,r)$, named for the Hungarian mathematician Pál Turán. Let $V_1, V_2, \ldots, V_r$ be a collection of pairwise disjoint, non-empty subsets of $V(G)$ such that $\cup_{1\leq i\leq r} V_i = V(G)$ and $||V_i| - |V_j|| \leq 1$ for all $1 \leq i, j \leq r$. This collection $\{V_i\}_{1\leq i\leq r}$ is called *almost-balanced partition* of $V(G)$. We say $\{V_i\}_{1\leq i\leq r}$ is *balanced* if $|V_i| = |V_j|$ for all $1 \leq i \leq r$.

Definition 4 (Turán graph) *The Turán Graph $T(n,r)$ is a graph on n vertices for which there exists an almost-balanced partition of its vertex set $V(T(n,r))$ into $\{V_i\}_{1\leq i\leq r}$, such that $E(T(n,r)) = \{xy : x \in V_i, y \in V_j, \forall\ i \neq j\}$. I.e. $T(n,r)$ is a complete r-partite graph on n vertices such that the partitions are as equal as possible.*

The Turán graph, $T(n,r)$ is actually the extremal example of a graph with the most edges that does not contain a K_{r+1}. A proof of which can be found in [Die05]. This then introduces us to a section of graph theory called extremal graph theory.

1.1.1 Extremal Graph Theory

Bollobás said in his book Modern Graph Theory [Bol98] that a quintessential extremal problem in extremal combinatorics is that of the forbidden subgraph problem: Given a graph F determine the number of edges a graph G can have until it must contain a subgraph isomorphic to F. In this way it generalizes the concept that Turán introduced with his graphs and results.

First we formalize the results of Turán and then draw similarities to our result.

We now will focus on similar results the case of hypergraphs. For the remainder of this subsection we will use the following notation. We say an r-graph F is *forbidden* in an r-graph G if there does not exist a copy of F as a subgraph of G. We denote $\mathcal{F}$ to be the family of

r-graphs that we wish to forbid in a larger graph G. An r-graph G is said to be $\mathcal{F}$*-free* if G does not contain a subgraph isomorphic to any member of $\mathcal{F}$. Let n be a positive integer, $\mathcal{G}$ be a collection of $\mathcal{F}$-free r-graphs on $l \leq n$ vertices. Note that in most cases we would like n to be orders of magnitudes larger than l.

Definition 5 (Turán number) *Let $\mathcal{F}$ be a family of forbidden r-graphs and let $\mathcal{G}$ be the collection of $\mathcal{F}$-free r-graphs, G, on n vertices. The Turán number* $\mathrm{ex}(n, \mathcal{F})$ *is defined to be*

$$\mathrm{ex}(n, \mathcal{F}) := \max\{|E(G)| : G \in \mathcal{G}\}.$$

Then $\mathrm{ex}(n, \mathcal{F})$ is the maximum number of edges in an $\mathcal{F}$-free r-graph of order n. Note that a 2-graph is simply a graph. A generalized topi this is to find the maximum number of copies of an r-graph A in a larger $\mathcal{F}$-free r-graph G. Note that an edge in an r-graph is still an r-graph in its own right so this idea generalizes Turán numbers. Then we have the following definition:

Definition 6 (Generalized Turán number) *Let $\mathcal{F}$ be a family of forbidden r-graphs and let $G \in \mathcal{G}$ be an $\mathcal{F}$-free r-graph on n vertices. Furthermore, let A be an $\mathcal{F}$-free r-graph on $k \leq n$ vertices and $C_A(G)$ be the set of all k-element subsets $V \subset V(G)$ such that $G[V]$ is isomorphic to A. Define the generalized Turán number* $\mathrm{ex}_A(n, \mathcal{F})$ *to be*

$$\mathrm{ex}_A(n, \mathcal{F}) := \max\{|C_A(G)| : G \in \mathcal{G}\}.$$

Now for our purposes we are going to eventually let n go to infinity so we must alter this definition slightly. First we define the edge density and the induced density of $H \in \mathcal{H}$ in a large G (generalized edge density) in the natural way

$$\deg(G) := \frac{|E(G)|}{\binom{n}{r}} \qquad d_H(G) := \frac{|C_A(G)|}{\binom{n}{k}}.$$

This allows us to rephrase the questions of "How many edges can a graph not containing a triangle have?" to a question of "What is the density of edges in a triangle-free graph?".

Now we define the Turán density.

Definition 7 (Turán density)

$$\pi(\mathcal{F}) := \lim_{n\to\infty} \frac{|E(G)|}{\binom{n}{r}}$$
$$\pi_H(\mathcal{F}) := \lim_{n\to\infty} \frac{|C_A(G)|}{\binom{n}{k}}$$

First we must show that these limits exist. An elegant proof of the first definition was given by Keevash [Kee11] and is included here for completeness.

Proposition 8 (Keevash) *The Turán density $\pi(\mathcal{G})$ exists.*

Proof. Let G be an r-graph with $\deg(G) = \mathrm{ex}(n, \mathcal{F})$. Denote $\psi(G) = \mathrm{ex}(n, \mathcal{F})/\binom{n}{r}$. Observe that $|E(G)| = |E(G - \{v\})| + \deg(v)$ for any $v \in V(G)$ so we may rewrite $\psi(G)$ as

$$\psi(G) = \frac{1}{n} \sum_{v\in V(G)} [|E(G - \{v\})| + \deg(v)] \cdot \binom{n}{r}^{-1}.$$

Replacing $|E(G - \{v\})|$ with $\psi(G - \{v\}) \cdot \binom{n-1}{r}^{-1}$ we get

$$\psi(G) = \frac{1}{n} \sum_{v\in V(G)} [\psi(G - \{v\}) \cdot \binom{n-1}{r}^{-1} + \deg(v)] \cdot \binom{n}{r}^{-1}.$$

Rearranging and doing some arithmetic gives

$$\psi(G) = \left[\frac{1}{n} \sum_{v\in V(G)} \psi(G - \{v\}) \cdot \binom{n-1}{r}^{-1}\right] + \left[\frac{1}{n} \sum_{v\in V(G)} \deg(v) \cdot \binom{n}{v}^{-1}\right].$$

By the handshaking lemma found in Modern Graph Theory [Bol98] for r-graphs we get $r \cdot |E(G)| = \sum_{v\in V(G)} \deg(v)$. Then

$$\psi(G) = \left[\frac{1}{n} \sum_{v\in V(G)} \psi(G - \{v\}) \cdot \binom{n-1}{r}^{-1}\right] + \left[\frac{r}{n}|E(G)| \cdot \binom{n}{v}^{-1}\right].$$

The definition of $\psi(G)$ gives the following recursion

$$\begin{aligned}\psi(G) &= \frac{1}{n} \sum_{v\in V(G)} \psi(G - \{v\}) \left(1 - \frac{r}{n}\right) + \frac{r}{n}\psi(G) \\ &= \frac{1}{n} \sum_{v\in V(G)} \psi(G - \{v\}).\end{aligned}$$

From the definition of $\mathrm{ex}(n-1,\mathcal{F})$ we have that $\frac{1}{n}\sum_{v\in V(G)}\psi(G-\{v\}) \leq \mathrm{ex}(n-1,\mathcal{F})\binom{n-1}{r}^{-1}$. Hence

$$\mathrm{ex}(n,\mathcal{F})\binom{n}{r}^{-1} \leq \mathrm{ex}(n-1,\mathcal{F})\binom{n-1}{r}^{-1}.$$

Therefore $\mathrm{ex}(n,\mathcal{F})$ is a decreasing sequence which is bounded below by zero. Hence $\pi(\mathcal{F})$ must exist. $\square$

Now the question remains as to whether we can find $\pi(F)$. For 2-graphs Turán himself found the result of $\pi(K_k) = 1 - \frac{1}{k}$ for all possible k. He even determined the graph that attains this value for each n. This is the Turán graph $T(n,k)$ we introduced earlier. But for all but a few other values of $k > r \geq 3$ we have no solution. Paul Erdős offered a reward, as was his custom, for solving this problem, a \$500 for any value of k and \$1000 if anyone can solve it for all k. For those familiar with his rewards system, these are fairly hefty rewards giving credence to the difficulty of these problems.

This however is only the groundwork for our results. We approach a similar problem in poset theory.

1.2 Poset Theory

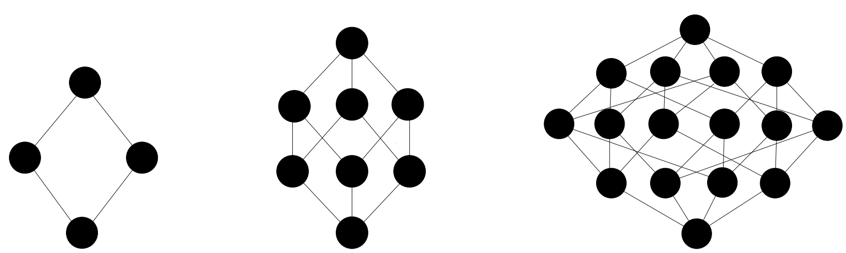
First we need to establish some definitions.

Definition 9 (Partially ordered set (Poset)) *A partial order is a binary relation "$\leq_{\mathcal{P}}$" over a set P which is reflexive, antisymmetric, and transitive, i.e. which for all $a,b,c \in \mathcal{P}$ satisfies:*

- $a \leq_{\mathcal{P}} a$ *(reflexivity)*
- *if* $a \leq_{\mathcal{P}} b$ *and* $b \leq_{\mathcal{P}} a$ *then* $a = b$ *(antisymmetry)*
- *if* $a \leq_{\mathcal{P}} b$ *and* $b \leq_{\mathcal{P}} c$ *then* $a \leq_{\mathcal{P}} c$ *(transitivity)*

A partially ordered set (Poset) is a set with a partial order.

Note that we will use "$\leq$" for "$\leq_{\mathcal{P}}$" when the binary operation is clear from context, similarly, we use $\mathcal{P}$ to denote its underlying set, when the context is clear. We define the operation $<$ such that for $A, B \in \mathcal{P}$, $A < B$ if $A \leq B$ and $A \neq B$. Two sets $A, B \in \mathcal{P}$ are said to be *comparable* if either $A \leq B$ or $B \leq A$, otherwise they are said to be *incomparable.* A partial order under which every pair of elements is comparable is called a *total order* or *linear order.* A set $\mathcal{C} \subset \mathcal{P}$ that is totally ordered is called a *chain.* A chain of order k will be denoted $\mathcal{C}_k$. A maximal chain is called a *full chain.* A set $\mathcal{A} \subset \mathcal{P}$ such that for all $A, B \in \mathcal{A}$, A and B are incomparable, is called an *antichain.* The *order* of a poset $|\mathcal{P}|$ is the cardinality of the set $\mathcal{P}$. An element $A \in \mathcal{P}$ is said to *cover* an element $B \in \mathcal{P}$ if $B \leq A$ and there does not exist an element $C \in \mathcal{P}$, $C \neq A, B$ such that $A \leq C \leq B$. An *interval* in a poset $\mathcal{P}$ is defined by $[A, B] = \{C \in \mathcal{P} : A \leq C \leq B\}$. The *height* of a poset $\mathcal{P}$, denoted $h(\mathcal{P})$ is defined to be k where k is the length of the longest chain in $\mathcal{P}$. Denote the *Hasse diagram* of a poset $\mathcal{P}$ to be the graph $G = (V, E)$ where V is the elements of $\mathcal{P}$ and $AB \in E$ if an only if A covers B or B covers A.



(a) $\mathcal{B}_2$ (b) $\mathcal{B}_3$ (c) $\mathcal{B}_4$

Figure 1.2: Examples of Hasse Diagrams

An element $A \in \mathcal{P}$ is said to be the *greatest (least) element* if for all other elements $B \in \mathcal{P}$, $B < A$ $(A < B)$. A *maximal (minimal)* element of $\mathcal{P}$ is an element $A \in \mathcal{P}$ such that there does not exist another element $B \in \mathcal{P}$, $A \neq B$ such that $A \leq B$ $(B \leq A)$. For a subset $\mathcal{B} \subseteq \mathcal{P}$ an element $A \in \mathcal{P}$ is said to be an upper (lower) bound of $\mathcal{B}$ if for all elements $B \in \mathcal{B}$, $B \leq A$ $(A \leq B)$. A poset, $\mathcal{P}$, is called a *lattice* if for every $A, B \in \mathcal{P}$ there exists elements $C, D \in \mathcal{P}$ where $C \leq A, B$ and $A, B \leq D$ and there does not exist $C', D' \in \mathcal{P}$ such that $C \leq C' \leq A, B$ and $A, B \leq D' \leq D$. Here C is often called the *infimum*, *greatest upper bound* or *meet* of A, B

and D is called the *supremum*, *least upper bound*, or *join* of A, B. Note that meet and join come from the algebraic construction of a lattice which is given a set S and two binary operations, $\vee$ and $\wedge$ called join and meet respectively, which are commutative, associative and idempotent such that for $A, B \in S$, $A \wedge (A \vee B) = A \vee (A \wedge B) = A$ (called the *absorption law* or *absorption identity*). Note that Huntington [Hun33] and McCune [McC97] proved equivalent statements to the absorption law in a lattice.

Some examples of partially ordered sets include:

- The real numbers (or subset thereof) ordered by the standard "less than or equal to". Note that this is a lattice, total order and an poset of infinite order. Also note that the standard "less than" does not form a poset since it would fail reflexivity.

- The natural numbers ordered by divisibility. Note that this is a lattice but not a total order (divisibility conditions and uniqueness of prime factors) and is also a poset of infinite order.

- For a given space X and a partially ordered set $\mathcal{P}$, the function space, F, containing all functions from X to $\mathcal{P}$ is a partially ordered set with the order for all $x, y \in X$, $x \leq_F y$ if and only if $f(x) \leq_{\mathcal{P}} f(y)$.

- The set $[n] := \{1, 2, \ldots, n\}$ along with all subsets of $[n]$ ordered by inclusion is a poset of finite order and is a lattice. This is a special lattice called the Boolean lattice in which we will take special interest in later.

We will refer to special posets by name throughout the rest of the dissertation. Recall that $\mathcal{C}_k$ will denote the chain of order k. The *r-fork* poset, denoted V_r, which has elements $A, B_1, \ldots, B_r$ such that $A \subset B_i$ for all i with no implied relation between B_i and B_j for any $1 \leq i, j \leq r$, $i \neq j$. A *broom* poset will consist of elements a chain of length k with the top set replaced with an r-fork. A *baton* poset will be a broom-like poset with the bottom element replaced with an inverted r-fork (note the top and bottom forks need not have the same value for r). The *butterfly* poset will consist of elements A, B, C, D such that $A, B \subset C, D$ and no other inclusions. The *complete bipartite* poset, $\mathcal{K}_{s,t}$ consists of elements A_i, B_j where

$1 \leq i \leq s, 1 \leq j \leq t$ and $A_i \subset B_j$ for all i, j. The $\mathcal{N}$-*poset* consists of elements A, B, C, D such that $A \subset C$ and $B \subset C, D$. The k-*crown* poset, $\mathcal{O}_{2k}$, $k \geq 2$ is the height 2 poset where the Hasse diagram is an undirected cycle on $2k$ vertices. The k-*diamond* poset $\mathcal{D}_k$ will consist of the elements $A, B_1, \ldots, B_k, C$ such that $A \subset B_i \subset C$ for all i. If $i = 2$ we will refer to this as the diamond poset denoted by $\mathcal{D}$. Note that $D \cong \mathcal{B}_2$.

As an aside the $\mathcal{N}$-poset is directly related to series parallel posets, see Valdes, Tarjan, and Lawler [VTL82]. These have direct application to many topics in computer science including sequencing events in time series data, see Mannila and Meek [MM00]; transmission sequencing requirements of multimedia presentations, see Amer, Chassot, Connolly, Diaz and Conrad [ACC$^+$94]; and to model the task dependencies in a dataflow model of massive data processing for computer vision, see Choudhary, Narahari, Nicol and Simha [CNNS94].

Now we give a series of definitions that lead up to the isomorphism definition for posets.

Given two posets $(\mathcal{U}, \leq_{\mathcal{U}})$ and $(\mathcal{T}, \leq_{\mathcal{T}})$ a function $f : \mathcal{U} \to \mathcal{T}$ is called *order-preserving* or *monotone* if for all $x, y \in \mathcal{U}$, $x \leq_{\mathcal{U}} y$ implies that $f(x) \leq_{\mathcal{T}} f(y)$. A function, f, is called *order-reflecting* if for all $x, y \in \mathcal{U}$, $f(x) \leq_{\mathcal{T}} f(y)$ implies that $x \leq_{\mathcal{U}} y$. Note that order-reflecting implies that f is injective since $f(x) = f(y)$ would imply that $x \leq_{\mathcal{U}} y$ and $y \leq_{\mathcal{U}} x$. If f is both order-preserving and order-reflecting f is called an *order-embedding* of $(\mathcal{U}, \leq_{\mathcal{U}})$ into $(\mathcal{T}, \leq_{\mathcal{T}})$. Note that if $|\mathcal{U}| < |\mathcal{T}|$ we consider $\mathcal{U}$ to be a *subposet* of T i.e. there exists an order-preserving function f from $\mathcal{U}$ to $\mathcal{T}$.

Definition 10 (Poset isomorphism) *Two posets, $\mathcal{P}$ and $\mathcal{P}'$, are isomorphic, denoted by $\mathcal{P} \cong \mathcal{P}'$ if there exists a bijective function f that is order-embedding, i.e., for all $A, B \in \mathcal{P}$, $A \leq_{\mathcal{P}} B$ if and only if $f(A) \leq_{\mathcal{P}'} f(B)$.*

Of special interest in this dissertation is the Boolean lattice of order n denoted $\mathcal{B}_n$ given in Example 1.2 above. $\mathcal{B}_n$ is also denoted $\mathcal{Q}_n$ in other literature, see Axenovich, Manske and Martin [AMM11]. Our results and almost all related results given henceforth will occur within the Boolean lattice. We will now proceed with the definitions we need to formulate our results. Most of these definitions are analogous to graph theory definitions. For clarity we will use capital letters when talking about sets in $\mathcal{B}_n$ to avoid confusing the sets with the values in $[n]$

itself.

Consider a family $\mathcal{F}$ of subsets of $[n]$. Then $\mathcal{F}$ can be viewed as a subposet of $\mathcal{B}_n$ with the inherited relation. Let $\mathcal{P}$ be a poset. If $\mathcal{F}$ contains no $\mathcal{P}$ as a subposet we say $\mathcal{F}$ is $\mathcal{P}$-free. Let $\mathrm{La}(n, \mathcal{P})$ be the the size of the largest family $\mathcal{F}$ of elements of $\mathcal{B}_n$ that is P-free. A *layer* of $\mathcal{B}_n$ is the family of all sets of the same cardinality, k, denoted $\binom{[n]}{k}$ for some integer k. Let $\mathcal{B}(n, k)$ denote the collection of subsets of the k middle sizes, that is, the collection of sets of size $\lfloor (n-k+1)/2 \rfloor, \cdots, \lfloor (n+k-1)/2 \rfloor$ or $\lceil (n-k+1)/2 \rceil, \cdots, \lceil (n+k-1)/2 \rceil$ depending on the parity of n. It is straightforward to see the connection here to Turán's graph and the forbidden subgraph problem.

There are many results concerning posets and we will not be able to reference all of them. We will only concern ourselves with papers related to our results and a few others. One area directly related to our results is forbidden induced posets. There are very few results and only results on small posets. See Joret and Milans [JM11] concerning two induced incomparable chains. Other results on two induced incomparable chains are Felsner, Krawczyk and Trotter [FKT13]. These are directly related to games on posets see Felsner, Kant, Pandu and Wagner [FKPRW00]. Other games on posets includes Bosek, Felsner, Kloch, Krawczyk, Matecki and Micek [BFK$^+$12] and Cranston, Kinnersley and Milans [CKM$^+$12]. Other popular topics in posets and that is beginning to be more closely tied to forbidden subposets are topics concerning the dimension of a graph, see Felsner and Trotter [FT05], adjacency posets, see Felsner, Li and Trotter [FLT10], defining lattice structures using planar graphs and other graphs , see Felsner [Fel04] and Felsner and Knauer [FK08]. This is not nearly an exhaustive list as it does not include any results in enumerative combinatorics which is very much alive in poset theory. These papers show that extremal combinatorics is also very interested in posets.

We now return our topic of forbidden (non-induced) posets. Sperner [Spe28] proved that the largest family of subsets of $\mathcal{B}_n$ with the property that no one of these sets contains another has size $\binom{n}{\lfloor n/2 \rfloor}$. Note that this is exactly the size of the largest layer in $\mathcal{B}_n$. Sperner's original proof involved using algebraic methods. It can also be derived from the YBLM (or LYM) inequality which was proved independently by Yamamoto [Yam54], Bollobás [Bol65], Lubell [Lub66] and Mešalkin [Meš63]. The proof is short but will be required for our results in Chapter 3 and is included for completeness.

Lemma 11 (YBLM Inequality) *If $\mathcal{A}$ is an antichain in $\mathcal{B}_n$ then*

$$\sum_{A\in\mathcal{A}} \binom{n}{|A|}^{-1} \leq 1.$$

Proof. All of the observations in this proof come from simple counting arguments.

Observe that the number of full chains in $\mathcal{B}_n$ is $n!$. Consider the chains that pass through each member of $\mathcal{A}$. Then for each $A \in \mathcal{A}$ the number of chains that pass through A is exactly $|A|!(n-|A|)!$. Hence the number of chains that pass through all members of $\mathcal{A}$ is $\sum_{A\in\mathcal{A}} |A|!(n-|A|)!$. Since no chain may contain two members of $\mathcal{A}$ (since it is an antichain) this sum must be less than the total number of chains in $\mathcal{B}_n$ or $\sum_{A\in\mathcal{A}} |A|!(n-|A|)! \leq n!$. Dividing by $n!$ on both sides gives the result. This completes the proof. □

Erdős [Erd45] generalized this result to show that, for any $k \geq 2$, a $\mathcal{C}_k$-free family of $\mathcal{B}_n$ has size at most $(k-1)\binom{n}{\lfloor n/2 \rfloor}$ asymptotically. Furthermore the size is exactly the sum of the $k-1$ middle layers. This result is given in the following theorem.

Theorem 12 (Erdős) *For $n \geq k-1 \geq 1$, $\mathrm{La}(n, \mathcal{C}_k) = \sum(n, k-1)$. Moreover, the $\mathcal{C}_k$-free families of maximum size in $\mathcal{B}_n$ are given by $\mathcal{B}(n,k)$.*

Other results include Katona and Tarján [KT83] which gives the result on 2-forks which was generalized by De Bonis and Katona [DBK07] to the following result:

$$\left(1 + \frac{r-1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq \frac{\mathrm{La}(n, \mathcal{V}_r)}{\binom{n}{\lfloor n/2 \rfloor}} \leq \left(1 + 2\frac{r-1}{n} + O\left(\frac{1}{n^2}\right)\right).$$

From this we can see that $\mathrm{La}(n, \mathcal{V}_r) \sim \binom{n}{\lfloor n/2 \rfloor}$. Thanh [Tha98] gave results on broom-like posets which was expanded upon by Griggs and Lu [GL09] to baton posets. De Bonis, Katona and Swanepoel [DBKS05] then showed that the butterfly poset, $\mathcal{B}$ has $\mathrm{La}(n, \mathcal{B}) \sim 2\binom{n}{\lfloor n/2 \rfloor}$. All the results up till now had the asymptotic form $k\binom{n}{\lfloor n/2 \rfloor}$ where k is one less than the height of the poset. De Bonis and Katona [DBK07] also gave results for the completely bipartite poset $\mathcal{K}_{s,t}$ which is a generalized butterfly to have $\mathrm{La}(n, \mathcal{K}_{s,t}) \sim 2\binom{n}{\lfloor n/2 \rfloor}$. It is of some note that for the butterfly and $\mathcal{K}_{s,t}$ posets that the lower bound is given by taking adjacent layers.

This construction fails if you take the two layers far enough apart. Thus, in a sense these constructions are unstable. The $\mathcal{N}$-poset, denoted $\mathcal{N}$, was shown by Griggs and Katona [GK08] to have $\mathrm{La}(n, \mathcal{N}) \sim \binom{n}{\lfloor n/2 \rfloor}$.

This led to Conjecture 13 by Griggs and Lu.

Conjecture 13 (Griggs-Lu) *Let $\mathcal{P}$ be any poset. Then $\pi(\mathcal{P}) := \lim_{n\to\infty} \frac{\mathrm{La}(n,\mathcal{P})}{\binom{n}{\lfloor n/2 \rfloor}}$ exists and is an integer.*

This was immediately generalized to Conjecture 14 by Saks and Winkler,

Conjecture 14 (Saks-Winkler) *Let $\mathcal{P}$ be any configuration. Then $\pi(\mathcal{P})$ is the the maximum number of full, adjacent layers that do not contain $\mathcal{P}$.*

Note that this last conjecture was not published by them in any journal but was communicated by Griggs, Li and Lu [GLL12].

This conjecture was upheld by Griggs and Lu [GL09] who proved that the k-crown poset for $k \geq 7$, k odd had an asymptotic value of $\binom{n}{\lfloor n/2 \rfloor}$, and later Lu [Lu14] proved that this is also true when $k \geq 4$ is even. This leaves only the 3- and 5-crown unsolved. Bukh [Buk09] proved the conjecture for posets, $\mathcal{T}$ whose Hasse diagram is a tree and found that $\mathrm{La}(n, \mathcal{T}) \sim h(\mathcal{T})\binom{n}{\lfloor n/2 \rfloor}$.

This may lead one to believe that the integer may depend on the height of the poset but this is not true. Griggs, Li and Lu [GLL12] proved that for the generalized diamond $\mathcal{D}_k$ the upper bound depends on the value of k not upon the height, $h(\mathcal{D}_k) = 3$. Let $m := \lceil \log_s(k + 2) \rceil$. They proved that if $k \in [2^{m-1} - 1, 2^m - \binom{m}{\lfloor m/2 \rfloor} - 1]$ then $\mathrm{La}(n, D_k) \sim m\binom{n}{\lfloor n/2 \rfloor}$. Note that the values of k include almost all the powers of 2^m, except 2 itself. The case of $k = 2$ is what we refer to as the diamond.

For many subposets $\mathcal{P}$, the size of the largest $\mathcal{P}$-free family is not known, even asymptotically. The smallest example of such is the diamond which is of special interest to us. Other small unsolved cases involve the 3- and 5-crown. We place an upper bound on the size of the largest 3-crown free family in Chapter 4. For now we will discuss other results pertaining to the diamond.

It is easy to see that if we take the two largest layers in $\mathcal{B}_n$ we will not contain the diamond. Also note that $\mathcal{D}$ is a subposet of $\mathcal{C}_4$ and hence the largest family has less than the size of the 3 middle layers (since this is the size of the largest family without a $\mathcal{C}_4$). From here every result uses what is called the Lubell function, to be defined next, to obtain their results.

Definition 15 *If $\mathcal{F}$ is a family of sets in the n-dimensional Boolean lattice, the **Lubell function** of that family is defined to be* $\Lambda(n, \mathcal{F}) \stackrel{\text{def}}{=} \sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1}$.

One way of describing $\Lambda(n, \mathcal{F})$ is that it calculates the average number of times a full chain in $\mathcal{B}_n$ contains an element of $\mathcal{F}$. However, in the transition to the Lubell function we lose much of the information given in $\mathcal{F}$. For instance, we lose much of the knowledge on how the sets in $\mathcal{F}$ interact. In order to compensate for the loss of this knowledge we will transition to asymptotic behavior. First we define needed terms.

Definition 16 *Let $\Lambda^*(n, \mathcal{P})$ be the maximum of $\Lambda(n, \mathcal{F})$ over all families $\mathcal{F}$ that are both $\mathcal{P}$-free and contain the empty set. Furthermore, set*

$$\Lambda^*(\mathcal{P}) \stackrel{\text{def}}{=} \limsup_{n \to \infty} \{\Lambda^*(n, \mathcal{P})\}.$$

We now provide a needed lemma.

Lemma 17 *Let $\mathcal{B}_2$ denote the diamond. If $\mathcal{F}$ is $\mathcal{B}_2$-free in $\mathcal{B}_n$ then $|\mathcal{F}| \leq (\Lambda^*(\mathcal{B}_2) + o(1))\binom{n}{\lfloor n/2 \rfloor}$. That is,* $\mathrm{La}(n, \mathcal{B}_2) \leq (\Lambda^*(\mathcal{B}_2) + o(1))\binom{n}{\lfloor n/2 \rfloor}$.

This lemma is restated and proven in 3.1.

A naive approach to the diamond problem using the Lubell function gives an upper bound of $\Lambda^*(n, \mathcal{B}_2) \leq 2.5$. Refining this idea, Griggs and Lu provided a simple proof in [GLL12] which improved this bound to $\Lambda^*(n, \mathcal{B}_2) \leq 2.296$. Then Axenovich, Mankse and Martin [AMM11] applied a new approach to reduce the bound to $\Lambda^*(n, \mathcal{B}_2) \leq 2.283$. Griggs, Li and Lu [GLL12] then refined their approach to get $\Lambda^*(n, \mathcal{B}_2) \leq 2.273$ using a much more careful analysis than what was involved in the 2.296 bound.

Our main result, also found in the paper by Kramer, Martin, and Young [KMY13] is

Theorem 18 (K-Martin-Young) *Let $\mathcal{F}$ be a diamond-free poset in the n-dimensional boolean lattice, $\mathcal{B}_n$. Then,*

$$|\mathcal{F}| \leq (2.25 + o(1))\binom{n}{\lfloor n/2 \rfloor}.$$

This result will be restated and proven in Chapter 3 but the proof is supplemented by the use of flag algebras even though knowledge of flag algebras are not required to follow it. We provide some background for flag algebras in Chapter 2.

This result is tight for known techniques however, since we are required to have $\emptyset \in \mathcal{F}$ for $\Lambda^*(n, \mathcal{B}_2)$. There are at least two known non-isomorphic families $\mathcal{F}$ that contain the empty set and give $\Lambda^*(n, \mathcal{B}_2) = 2.25$. They are given in Griggs, Li and Lu [GLL12] as follows:

Example 19

- *Consider the family of sets* $\mathcal{S} = \{\emptyset\}$, $\mathcal{T} = \{\{i\} : i \leq \lfloor n/2 \rfloor\}$ *and* $\mathcal{U} = \{\{i,j\} : \text{ either } i > \lfloor n/2 \rfloor \text{ or } j > \lfloor n/2 \rfloor\}$. *Let* $\mathcal{F} = \mathcal{S} \cup \mathcal{T} \cup \mathcal{U}$.

- *Consider the family of sets* $\mathcal{S} = \{\emptyset\}$, $\mathcal{T} = \{\{i,j\} : i > \lfloor n/2 \rfloor \text{ and } j > \lfloor n/2 \rfloor\}$, $\mathcal{T}' = \{\{i,j\} : i \leq \lfloor n/2 \rfloor \text{ and } j \leq \lfloor n/2 \rfloor\}$, $\mathcal{U} = \{\{i,j,k\} : i,j \leq \lfloor n/2 \rfloor \text{ and } k > \lfloor n/2 \rfloor\}$ *and* $\mathcal{U}' = \{\{i,j,k\} : i,j > \lfloor n/2 \rfloor \text{ and } k \leq \lfloor n/2 \rfloor\}$. *Let* $\mathcal{F} = \mathcal{S} \cup \mathcal{T} \cup \mathcal{T}' \cup \mathcal{U} \cup \mathcal{U}'$.

For either case we have that $\Lambda^*(n, \mathcal{B}_2) = 2.25$. Therefore our theorem is the best possible under current approaches.

Before we move on to flag algebras we have a few more minor definitions that are required.

1.3 Some Matrix Terminology and Definitions

There will be no discussion on these definitions. They may seem arbitrary but are necessary for our discussion on flag algebras.

Definition 20 (Positive Semidefinite Matrix) *A* positive semidefinite matrix *is a matrix A such that A is Hermitian and for all* $\mathbf{x} \neq 0$ $x^* A x \geq 0$.

For our purposes we will need a real valued matrix so this definition becomes

Definition 21 (Real Positive Semidefinite Matrix) *A* real positive semidefinite matrix *is a matrix A such that A is symmetric and for all* $\mathbf{x} \neq 0$ $x^T A x \geq 0$.

Let S_n be the collection of positive semidefinite matrices. If $X \in S_n$ then will also be denoted by $X \succeq 0$. Let $T : S_n \to \mathbb{R}^m$ be a linear transformation. Let $\mathbf{b} \in \mathbb{R}^m$ be a vector and $C \in S_n$ be given. A standard linear semidefinite program is of the form

$$\begin{array}{rr} \min & \operatorname{tr} CX \\ \text{subject to} & TX = \mathbf{b} \\ & X \succeq 0 \end{array}$$

For more information see Handbook of Linear Algebra [Hog07].

CHAPTER 2. Flag Algebras

Flag algebras was developed by Alexander Razborov [Raz07]. A survey paper was written by Razborov on the uses of flag algebras. The survey is found both on his personal web page [Raz] and also in the book *Mathematics of Paul Erdős II* [RLG13]. The results given in this survey include clique densities, Turán's tetrahedron problem, the Caccetta-Häggkvist conjecture, induced subgraph densities and hypergraph Ramsey problems. The first formal presentation of flag algebras occurs in finite model theory in its full generality but Razborov himself said that the creation of this method is geared toward extremal graph theory. For ease of reading we will extract the definitions as they pertain to extremal graph theory and our specific application.

Before we begin our discussion of flag algebras we must lay some basic groundwork in which our calculations will operate. For motivation we refer you back to the Subsection 1.1.1 entitled Extremal Graph Theory . If you recall, many questions in extremal graph theory can be phrased in such a way to be questions concerning densities. These densities may take several forms. The example we give is Mantel's Theorem which will be proven using flag algebras in Section 2.2. To lay the groundwork for this example this we offer the following definition.

Definition 22 *Fix a graph M of order l. For a graph G of order n let$D(M,G)$ be the number of subgraphs, U, in G such that U is isomorphic to M. Then define the following:*

$$d_n(M,G) = \frac{D(M,G)}{\binom{n}{l}},$$

$$d_n(M) = \max\{d_n(M,G) : |G| = n\}.$$

If we further suppose that $\lim_{n\to\infty} d_n(M)$ exists we define

$$\pi(M) = \lim_{n\to\infty} d_n(M).$$

Then $d_n(M,G)$ is the density of l-sets of G that are isomorphic to M, and $d_n(M)$ is the maximum value that $d_n(M,G)$ takes over all graphs G of order n. We will use $\pi(M)$ to denote our density in whatever setting we are in. Note that for $\pi(M)$ the limit is over n and the maximum is over G and hence $\pi(M)$ is not a lim sup. Therefore we are not guaranteed to have $\pi(M)$ exist. The existence of $\pi(M)$ would need to be verified or at least assumed to exist in order to ask what the value of $\pi(M)$ would be. In some cases like Mantel's Theorem, we can prove the existence of $\pi(M)$ before we calculate.

Now $d_n(M,G)$ and $d_n(M)$ can always be accomplished by a simple brute force methods for each n but this is tedious and an eternal quest if we should choose to attack the problem this way. A more efficient way would be to upper bound $d_n(M)$ in terms of n or l. One method would be to take the collection of graphs, $\mathcal{H}_k$ such that $H \subset G$, $|H| = k$, $k \geq l$ and calculate the density of M within each possible H. Then one can take a weighted average, dependent on the densities of the H within G, and get the density of M. To formalize this idea, fix $k \geq l$. Let $\mathcal{H}$ be the collection of all subgraphs of order k. It then follows that

$$d_n(M,G) = \sum_{H \in \mathcal{H}} d_k(M,H) d_n(H,G). \tag{2.1}$$

Note that $d_k(M,H)$ is not dependent on n since $|M| = l$ and $|H| = k$. We will capitalize on this. To achieve a basic upper bound we may find the H with the highest density of M and assume all other subgraphs $H \in \mathcal{H}$ have $d_n(H,G) = 0$. Then the summation becomes

$$d_n(M,G) \leq \max\{d_k(M,H) : H \in \mathcal{H}\}$$

This however is extremely naive and is often a very poor upper bound. One thing we have ignored in this construction is that most if not all pairs of $H, H' \in \mathcal{H}$ will interact in some way. For instance the density of the two subgraphs

$$H = K_{\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil} \quad H = K_{\lfloor \frac{k}{2} \rfloor + 1, \lceil \frac{k}{2} \rceil - 1}$$

would be very much related in most graphs. Taking this fact into consideration would potentially improve our bounds considerably. To facilitate this we use the approach of flag algebras. First we must begin with some basic definitions to lay the ground work for applying this method.

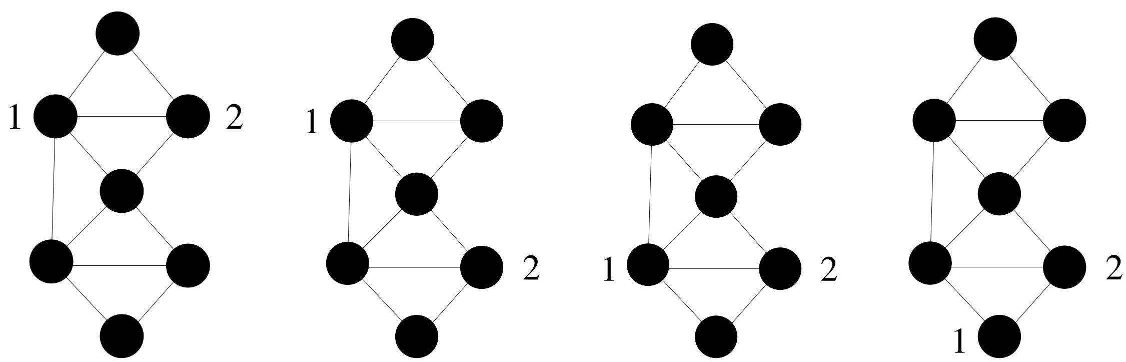
2.1 Definitions and Methods

Denote $[n] = \{1, \ldots, n\}$. We start with the basic definitions of a type and a σ-flag.

Definition 23 *A type, σ, is a graph J along with an injective function $\theta : [s] \to V(J)$ where $s = |V(J)|$.*

Definition 24 *Let K be a graph and $\theta : [s] \to V(K)$ be an injective function where $s \leq |V(K)|$. Given a type σ we say the ordered pair $F = (K, \theta)$ is a σ-flag if the graph induced on $\theta([s])$ is isomorphic to σ. I.e. The labeled part of K is isomorphic to σ. Let $\mathcal{F}$ be a family of forbidden graphs. We say the σ-flag F is admissible if K doesn't contain a member of $\mathcal{F}$ as a subgraph.*

We give examples of four σ-flags, where $s = 2$, in Figure 2.1. Note that the type is the labeled vertices and the flags are the entire graph given.



(a) σ-flag A (b) σ'-flag B (c) σ-flag C (d) σ-flag D

Figure 2.1: Examples of σ and σ'-flags with $\sigma = K_2$ and $\sigma' = \overline{K_2}$

We observe several things about Figure 2.1. The type of flag A, C, and D is an edge. The type of Flag B is a non-edge. Hence B cannot be isomorphic to A, C or D. Even though Flag D has an edge for a type, one of the vertices incident to this edge had degree two. Since all the vertices in Flags A and C have degree three, Flag B is not isomorphic to Flags A and C. Flags A and C are isomorphic.

Later we will find it useful to think of a type as simply the labeled parts of a subgraph of K. In this way we may look at several different σ-flags that contain the same type and be able to compare them. We now begin the formal definitions to accomplish this. To simplify matters we borrow notation from Razborov [Raz07].

Definition 25 *Given two σ-flags $F = (K, \theta)$ and $F' = (K', \theta')$ we say that F and F' are isomorphic ($F \cong F'$) if there exists a graph isomorphism $\tau : K \to K'$ such that the labeled graph on $\tau(\theta([s]))$ is isomorphic, including the labels, to the labeled graph on $\theta'([s])$. I.e. not only is τ an isomorphism between K and K', $\tau(\theta)$ and θ' induce the same labeled graph and are isomorphic to σ.*

Definition 26 *For a family of graphs $\mathcal{F}$, let $\mathcal{F}_m^\sigma$ be the collection of all admissible σ-flags of order m, up to isomorphism.*

Note that in Figure 2.1 flags A, C, and D belong to $\mathcal{F}_7^\sigma$ where σ has two vertices and an edge. Flag B belongs to $\mathcal{F}_7^{\sigma'}$ where σ' has two vertices and no edge.

Observe that for a specified σ, any two σ-flags $F = (K, \theta), F' = (K', \theta')$ by definition must immediately have that the types within F and F' be isomorphic. In fact we could rephrase this to say that we only need to concern ourselves with the automorphisms on σ that can be extended to isomorphisms on the graphs K and K'. In this way we may reduce the number of isomorphisms we need consider. I.e. we must only check to see if F and F' are isomorphic where the restricted isomorphism is one of the automorphisms on σ.

Note that it is quite possible to construct different flags with different types from the same graph K. We did so with the flags in Figure 2.1.

Now we give the definitions necessary to start comparing these different flags. First we define an index over all injections.

Definition 27 *Let σ be a type and G be a graph. Define Θ to be the set of all injections $\theta : [|\sigma|] \to V(G)$.*

Next we define a set of probabilities needed for the method.

Definition 28 *Let G be an $\mathcal{F}$-free graph of order $n \geq m$. Fix a σ-flag $F \in \mathcal{F}_m^\sigma$ and $\theta \in \Theta$. Define $p(F, \theta; G)$ to be the probability that a m-set W chosen uniformly at random from $V(G)$ such that $G[W]$ contains $\mathrm{Im}(\theta)$, induces a σ-flag $(G[W], \theta)$ isomorphic to F.*

Observe that these probabilities $p(F, \theta; G)$ are not just the probabilities that we find the subgraph F in G. These probabilities also induce a specific partial numbering of F. To see these are different take the complete bipartite graph $K_{n,n}$. Consider the type K_2 (two vertices with an edge). If we would label vertices from the different bipartition we may get a positive probability depending on the flag, but if we label two vertices from the same bipartition we must have a zero probability since there can never be an edge there.

This definition gives us individual probabilities based upon each individual flag and type. However we wish to compare flags to each other so we need a probability that would allow us to do this.

Definition 29 *Given σ-flags $F_a, F_b \in \mathcal{F}_m^\sigma$ and $\theta \in \Theta$, let W_a be an m-set from $V(G)$ chosen uniformly at random such that* $\mathrm{Im}(\theta) \subseteq \mathrm{W_a}$. *Also, let W_b be an m-set from $V(G)$ chosen uniformly at random such that* $\mathrm{Im}(\theta) = \mathrm{W_a} \cap \mathrm{W_b}$. *Define $p(F_a, F_b, \theta; G)$ to be the probability that both $(G[W_a], \theta) \cong F_a$ and $(G[W_b], \theta) \cong F_b$.*

Note the difference between $p(F_a, F_b, \theta; G)$ and $p(F_a, \theta; G)p(F_b, \theta; G)$. Baber and Talbot [BT11] described this difference as sampling without replacement and with replacement respectively. Due to a proof by Razborov [Raz07] we have Lemma 30, below. This result is quite intuitive. If we fix θ, W_a and W_b and then have $|V(G)| \to \infty$, the chance that $W_a \cap W_b$ contains anything other than θ becomes negligible since the odds of W_a and W_b choosing the same vertex not in θ goes to zero. We include a proof of this lemma for completeness.

Lemma 30 *For any $F_a, F_b \in \mathcal{F}_m^\sigma$ and $\theta \in \Theta$,*

$$p(F_a, \theta; G)p(F_b, \theta; G) = p(F_a, F_b, \theta; G) + o(1)$$

where $o(1) \to 0$ as $|V(G)| \to \infty$.

Proof.

Let $n := |V(G)|$ and W_a, W_b be two m-sets chosen uniformly at random from $V(G)$ such that both W_a and W_b contain θ. Note that this definition of W_a and W_b allows for additionally vertices in common between W_a and W_b than just the vertices in θ. Since the condition that

θ be in the image can be viewed as a conditional probability on both sides we may restrict our search to the vertices in $W' = W_a - \mathrm{Im}(\theta)$ and $W'' = W_b - \mathrm{Im}(\theta)$. It then follows that $W_a \cap W_b = \mathrm{Im}(\theta)$ if and only if $W' \cap W'' = \emptyset$.

For clarity we will transition notation for this proof as Razborov's notation is cumbersome when we are speaking strictly about probabilities. Let A be an event where $(G[W_a], \theta) \cong F_a$, B be an event where $(G[W_b], \theta) \cong F_b$, and C be an event where $W_a \cap W_b = \emptyset$. We will use $P(\cdot)$ to designate the probability of an event happening, mostly to differentiate from the flag notation.

Then have that

$$\begin{aligned} p(F_a, F_b, \theta; G) &= P(A \cap B | C) = \frac{P(A \cap B \cap C)}{P(C)} \\ p(F_a, \theta; G) &= P(A) \\ p(F_b, \theta; G) &= P(B) \end{aligned}$$

Observe that by our definition A and B are independent. Since $A \cap B \cap C \subset A \cap B$ we have that

$$P(A \cap B \cap C) \leq P(A \cap B) = P(A)P(B). \tag{2.2}$$

Also note that since C and $\overline{C}$ are disjoint we have that

$$\begin{aligned} P(A)P(B) = P(A \cap B) &\leq P(A \cap B \cap (C \cup \overline{C})) \\ &\leq P(A \cap B \cap C) + P(A \cap B \cap \overline{C}) \\ &\leq P(A \cap B \cap C) + P(\overline{C}). \end{aligned} \tag{2.3}$$

Combining Equations 2.2 and 2.3 gives

$$P(A \cap B \cap C) \leq P(A)P(B) \leq P(A \cap B \cap C) + P(\overline{C})$$

Returning to our flag algebra notation and doing some algebra we have that

$$0 \leq p(F_a, \theta; G)p(F_b, \theta; G) - p(F_a, F_b, \theta; G)P(C) \leq P(\overline{C})$$

All that is left is to show that $P(C) \to 1$ and $P(\overline{C}) \to 0$ as $n \to \infty$. Observe that $P(C) + P(\overline{C}) = 1$. Consider $P(C)$. Note that the number of ways to choose an m-set such that the exact vertices of θ is contained in it is $\binom{n-|\theta|}{m-|\theta|}$. Also note that the number of ways to choose a second m-set containing the vertices in θ that overlaps the first m-set exactly on the vertices of θ is $\binom{(n-|\theta|)-(m-|\theta|)}{m-|\theta|} = \binom{n-m}{m-|\theta|}$. Since the m-sets are chosen uniformly at random we have that

$$P(C) = \frac{\binom{n-|\theta|}{m-|\theta|}\binom{n-m}{m-|\theta|}}{\binom{n-|\theta|}{m-|\theta|}\binom{n-|\theta|}{m-|\theta|}} = \frac{\binom{n-m}{m-|\theta|}}{\binom{n-|\theta|}{m-|\theta|}}$$

Since m and $|\theta|$ are fixed $P(C) \to 1$ as $n \to \infty$. Hence $P(\overline{C}) \to 0$ as $n \to \infty$. Therefore as $n \to \infty$ we have that

$$p(F_a, F_b, \theta; G) \to p(F_a, \theta; G)p(F_b, \theta; G).$$

This completes the proof of Lemma 30. □

This result is quite useful in extremal results since it implies that we may ignore the cases where flags overlap. It also gives rise to our method of improving $p(M, G) \leq \max_{H \in \mathcal{H}}\{p(M, H)\}$. Recall that for any graphs G and M of order n and l respectively we have that $d_n(M, G) = \sum_{H \in \mathcal{H}} d_k(M, H)d_n(H, G)$ where $\mathcal{H}$ is the collection of graphs of order $k \geq l$ equation (2.1). Our goal is to construct a summation over $\mathcal{H}$ with coefficients c_H, $H \in \mathcal{H}$ such that

$$0 \leq \sum_{H \in \mathcal{H}} c_H d_n(H, G). \tag{2.4}$$

We will then combine Equations 2.1 and 2.4 to get

$$d_n(M, G) \leq \sum_{H \in \mathcal{H}} d_k(M, H)d_n(H, G) + \sum_{H \in \mathcal{H}} c_H d_n(H, G) = \sum_{H \in \mathcal{H}} (d_k(M, H) + c_H)d_n(H, G). \tag{2.5}$$

It then follows that

$$d_n(M, G) \leq \max\{d_k(M, H) + c_H : H \in \mathcal{H}\}.$$

Observe that in Equation 2.4 we only require that the sum be nonnegative; we require nothing about the individual values of c_H. In this way we can add negative numbers to high values of $p_n(M,H)$ and add positive numbers to low values of $p_n(M,H)$. Even though the choice we make can raise the total sum in Equation 2.1 we can still lower the maximum value of $p_n(M,H)$.

To guarantee that Equation 2.4 is nonnegative we will construct it from a square of linear sums of flags. i.e.

$$\begin{aligned} 0 &\leq \left(\sum_{F_i \in \mathcal{F}_m^\sigma} q_i p(F_i, \theta; G) \right)^2 \\ &= \sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_i q_j p(F_i, \theta; G) p(F_j, \theta; G) \\ &= \sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_i q_j p(F_i, F_j, \theta; G) + o(1). \end{aligned} \tag{2.6}$$

The values $q_i \in \mathbb{R}$ will allow us to adjust this sum to make the best choices for the values of c_H. However, the values of $p(F_i, F_j, \theta; G)$ can and most likely will become arbitrarily difficult to calculate as $n \to \infty$. To avoid this we will have to choose the order of our flags and the order of our graphs in $\mathcal{H}$ carefully. Let $2m - |\sigma| \leq k$. Then we will be able to allow two flags from $\mathcal{F}_m^\sigma$ would be allowed to overlap in a graph $H \in \mathcal{H}$ exactly on θ which is required if we want to use calculations found in Equation 2.6. Similar to the discussion for $d_n(M,G)$ we can adjust the Equation 2.6 as follows

$$\sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_i q_j p(F_i, F_j, \theta; G) = \sum_{H \in \mathcal{H}} \sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_i q_j p(F_i, F_j, \theta; H) d_n(H, G). \tag{2.7}$$

Here we abused notation slightly with the use of θ but since $p(F_a, F_b, \theta; H) = 0$, if θ is not contained within H, there is no ambiguity. This allows us to focus completely on H instead of G. (Note that the $d_n(H,G)$ matches the term in Equation 2.1 and will disappear when we switch to the maximum density.) It might be helpful to think of the graphs H as a sample of the graph G. Since we will cover all of G we can see how each sample behaves and be able to average them all together accurately. However, this is not enough to satisfy our calculations.

One thing we must remove from our calculations is the requirement of θ as it is not present in Equation 2.1. To do this define $p(F_a, F_b; H)$ to be the average value that $p(F_a, F_b, \theta; H)$ as θ ranges over Θ. We may then take the estimated value over Θ for Equation 2.7 and due to linearity of expectation we get

$$\begin{aligned}
&\mathbb{E}_{\theta\in\Theta}\left[\sum_{H\in\mathcal{H}}\sum_{F_i,F_j\in\mathcal{F}_m^\sigma} q_iq_jp(F_i,F_j,\theta;H)d_n(H,G)\right]\\
&=\sum_{H\in\mathcal{H}}\sum_{F_i,F_j\in\mathcal{F}_m^\sigma} q_iq_j\mathbb{E}[p(F_i,F_j,\theta;H)]d_n(H,G)\\
&=\sum_{H\in\mathcal{H}}\sum_{F_i,F_j\in\mathcal{F}_m^\sigma} q_iq_jp(F_i,F_j;H)d_n(H,G)
\end{aligned}$$

Note that since F_a, F_b and H are finite quantities that do not depend on n we can easily calculate $p(F_a, F_b; H)$ and apply it to any n, and most ideally as $n \to \infty$. This allows us to define

$$c_H = \sum_{F_i,F_j\in\mathcal{F}_m^\sigma} q_iq_jp(F_i,F_j;H).$$

Combining this with the basic idea that the estimated value of nonnegative values is nonnegative gives

$$\begin{aligned}
0 &\le \mathbb{E}\left[\sum_{F_i,F_j\in\mathcal{F}_m^\sigma} q_iq_jp(F_i,F_j;G)\right]+o(1)\\
&=\sum_{H\in\mathcal{H}}\sum_{F_i,F_j\in\mathcal{F}_m^\sigma} q_iq_jp(F_i,F_j;H)d_n(H,G)+o(1)\\
&=\sum_{H\in\mathcal{H}} c_Hd_n(H,G)
\end{aligned}$$

which satisfies our Equation 2.4 and gives us

$$d_n(M,G) \le \max\{d_k(M,H)+c_H : H\in\mathcal{H}\}.$$

All that is left to do is optimize our values q_iq_j to lower the max as much as possible. Our choice of notation hints at the process that we will use, namely using a semi-definite matrix to optimize these values.

Recall that if we take two symmetric matrices $A, B \in \mathbb{R}^{n\times n}$ we make define an inner product of A and B, which corresponds to the Frobenius norm, as

$$\langle A,B\rangle := \operatorname{tr}(A^TB) = \sum_{0\le i,j\le n} a_{ij}b_{ij}.$$

Showing that this is an inner product is an undergraduate linear algebra exercise and is omitted. If we define matrices $Q = [Q_{ij}] = [q_i q_j]$ and $P(H) = [p(F_i, F_j; H)]$, which are obviously symmetric, then we may rewrite c_H as

$$\begin{aligned} c_H &= \sum_{F_i, F_j \in \mathcal{F}_m^\sigma} q_i q_j p(F_i, F_j; H) \\ &= \langle Q, P \rangle . \end{aligned}$$

In this form these equations become the constraints to the minimization problem

$$d_n(M, G) \leq \inf_{Q \in \mathcal{Q}} \left\{ \sup_{\{d_n(H,G):H \in \mathcal{H}\}} \left\{ \sum_{H \in \mathcal{H}} (d_k(M, H) + c_H) d_n(H, G) + o(1) \right\} \right\}$$

where $\mathcal{Q}$ is the collection of all semidefinite matrices and c_H depends on Q. This is now in the form of a linear semidefinite program, see 1.3. Now a computer using an semidefinite program (SDP) solver can optimize this solution, with some error bar of course. Now we must also consider computationally viable our system is. If $|\sigma|, m$ or k is too large this computation will not complete in a reasonable amount of time. But keeping these values small is not the only way to speed along the computation.

If we are further willing to relax the conditions a bit we can make the computation easier to handle by only considering optimizing over the vector space $V = \{v = (d_k(M, H))_{H \in \mathcal{H}}\}$. Then we can use the vectors $\{v_H \in V : v_H(H') = 1_{H=H'}\}$ where $1_{H=H'}$ is the Kronecker delta function. This reduces the minimization problem to

$$\inf \left\{ \max\{d_k(M, H) + c_H \; H \in \mathcal{H}\} + o(1) : Q \in \mathcal{Q} \right\} .$$

Now if we assume $\pi(H)$ exists we may transition to the extremal case. In this setting the $o(1)$ term is irrelevant and our problem further reduces to

$$\pi(M) \leq \max\{d_k(M, H) + c_H : H \in \mathcal{H}\}.$$

This is our eventual goal for our problem and we give an example to emphasize that this is indeed feasible. In fact in the example below you can find the positive semidefinite matrices by hand.

2.2 Mantel's Thoerem

Theorem 31 is stated here in our context. An alternate statement and proof can be found in Modern Graph Theory [Bol98].

Theorem 31 (Mantel) *Let C_3 denote the cycle of length 3. Let*

$$\pi(K_2) := \lim_{n\to\infty} \left\{ \max \left\{ \frac{|E(G)|}{\binom{n}{2}} : |V(G)| = n, G \text{ contain no } C_3 \right\} \right\},$$

i.e. $\pi(K_2)$ is the limit as n goes to infinity of the maximum of the density of edges in any graph that forbids C_3. Then

$$\pi(K_2) = 1/2.$$

Observe that the complete bipartite graph $K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ contains $\lceil n/2 \rceil \lfloor n/2 \rfloor$ edges which immediately gives a asymptotic lower bound of $1/2$ to $\pi(K_2)$, i.e $\pi(K_2) \geq 1/2$.

This proof was first given by Falgas-Ravry and Vaughan [FRV12] as a demonstration of the use of flag algebras. We will adapt their proof to use our notation. This is an extremely simple example and avoids many of the issues of multiple σ and keeps the number of σ-flags and sample graphs H very low.

Proof of Theorem 31 Let $G = K_n$. Color the edges of G with two colors, red and blue. We may associate any graph G' on n vertices with a coloring of G by coloring the edge, E, red if $E \in E(G')$ and blue otherwise.

We let $\Theta = \{\sigma\}$ where $\sigma = K_1$, the single vertex. Let $m = 2$ and $\mathcal{F}_2^\sigma = \{F_0, F_1\}$ where F_0 is a red edge and F_1 is a blue edge, i.e. our flags are the graphs on two vertices with the edges colored red and blue respectively. Let $k = 3$ and let $\mathcal{H} = \{H_0, H_1, H_2\}$ where H_i is the graph on 3 vertices with exactly i red edges. Since G contains no K_3, $\mathcal{H}$ contains all of the colorings on 3-vertex subgraphs of G. Let Q be a positive semidefinite matrix such that the q_{ij} entry corresponds to the probability $p(F_i, F_j, \theta; G)$. I.e. for $l = 0, 1, 2$ we have

$$c_{H_k} = q_{00}p(F_0, F_0; H_l) + q_{01}p(F_0, F_1; H_l) + q_{10}p(F_1, F_0; H_l) + q_{11}p(F_1, F_1; H_l).$$

Now we calculate the values of $p(F_i, F_j; H_l)$ for each i, j, l. Since we are dealing with the average value over Θ we must consider all possibilities of θ. In this case it's fairly trivial, we just look at the values for each vertex in H. Now since there are only blue edges in H_0, $p(F_i, F_j; H_0) = 1$ if $i = j = 0$ and 0 otherwise. A less trivial example is H_1. Since in our calculations the choice of flags is ordered, for each choice of θ there are two ways flags can occur. Hence if θ will label the vertex not incident with any red edge we have the ordered pair of flags (F_0, F_0) will get counted twice. If θ will label a vertex incident to the red edge we have the ordered pairs of flags (F_0, F_1) and (F_1, F_0) each counted once. Since there are two possible ways to label a vertex incident to exactly one red edge the ordered pairs of flags (F_0, F_1) and (F_1, F_0) will each get counted twice in total. Similarly if we take H_2 we will get two ordered pairs of (F_1, F_1) if we label the vertex incident to both red edges and one ordered pair of both (F_0, F_1) and (F_1, F_0) for each of the vertices incident to only one red edge. There are no vertices incident to only blue edges so there are no copies of (F_0, F_0). Since there are 6 total possible choices for choosing θ and each ordered pair of flags, we divide by six to get the probability of getting any pair uniformly at random. You will find the probabilities associated each ordered pair of flags in Table 2.1.

	$p(F_0, F_0; H_l)$	$p(F_1, F_0; H_l)$	$p(F_0, F_1; H_l)$	$p(F_1, F_1; H_1)$
H_0	1	0	0	0
H_1	1/3	1/3	1/3	0
H_2	0	1/3	1/3	1/3

Table 2.1: Probabilities $p(F_i, F_j; H_l)$

Then the equations c_{H_l} are

$$\begin{aligned} c_{H_0} &= q_{00} \\ c_{H_1} &= \frac{1}{3}q_{00} + \frac{1}{3}q_{01} + \frac{1}{3}q_{10} \\ c_{H_2} &= \frac{1}{3}q_{01} + \frac{1}{3}q_{10} + \frac{1}{3}q_{11} \end{aligned}$$

Since density of red edges is our primary concern, let M be the two vertex graph with a red edge. Then it follows that $d_3(M, H_0) = 0$, $d_3(M, H_1) = 1/3$ and $d_3(M, H_3) = 2/3$. Let $q_{00} = 1/2$. Then $d_3(M, H_0) + c_{H_0} = 1/2$. Note that this choice was motivated by the desire to make $d_3(M, H_i) + c_{H_i} \leq 1/2$ for all $i = 0, 1, 2$. Since $q_{10} = q_{01}$ in a positive semidefinite matrix gives the matrix

$$Q = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

which gives that

$$\begin{aligned} d_n(M, G) \leq d_n(H_0, G)(d_3(M, H_0) + c_{H_0}) + d_n(H_1, G)(d_3(M, H_1) + c_{H_1}) \\ + d_n(H_2, G)(d_3(M, H_2) + c_{H_2}) + o(1) \\ = \frac{d_n(H_0, G)}{2} + \frac{d_n(H_1, G)}{6} + \frac{d_n(H_2, G)}{2} + o(1) \\ \leq \frac{d_n(H_0, G) + d_n(H_1, G) + d_n(H_2, G)}{2} + o(1) \\ = \frac{1}{2} + o(1). \end{aligned}$$

By the generality of G, and the squeeze theorem, we have that $\pi(K_2) = 1/2$. This concludes the proof of Mantel's Theorem. $\square$

CHAPTER 3. Main Results

In this section we discuss our approach to improve the bound on the diamond. The statement of Theorem 18 is stated again for convenience.

Theorem 18 (K-Martin-Young) *Let $\mathcal{F}$ be a diamond-free poset in the n-dimensional boolean lattice, $\mathcal{B}_n$. Then,*

$$|\mathcal{F}| \leq (2.25 + o(1))\binom{n}{\lfloor n/2 \rfloor}.$$

The proof of this theorem follows from three lemmas: Lemma 32, Lemma 35 and Lemma 40 which are each given their own section in this chapter. Lemma 32 places an upper bound on $|\mathcal{F}|$ using $\Lambda^*(\mathcal{B}_2)$, but there are examples that show that $\Lambda^*(\mathcal{B}_2) \geq 2.25$. We will discuss these in Section 3.1. Lemma 35 then converts our poset problem by upper bounding $\Lambda^*(\mathcal{B}_2)$ with a function f depending a graph generated by $\mathcal{F}$. This will be discussed in Section 3.2. Lemma 40 then upper bounds the function f by $.25 + o(1)$. This will be discussed in Section 3.3.

3.1 Placing an upper bound on $|\mathcal{F}|$ using $\Lambda^*(\mathcal{B}_2)$

The following lemma was originally found in Griggs and Lu [GL09].

Lemma 32 (Griggs-Lu) *Let $\mathcal{B}_2$ denote the diamond. If $\mathcal{F}$ is $\mathcal{B}_2$-free in $\mathcal{B}_n$ then $|\mathcal{F}| \leq (\Lambda^*(\mathcal{B}_2) + o(1))\binom{n}{\lfloor n/2 \rfloor}$. That is, $\mathrm{La}(n, \mathcal{B}_2) \leq (\Lambda^*(\mathcal{B}_2) + o(1))\binom{n}{\lfloor n/2 \rfloor}$.*

Proof. First observe that we may trivially bound the size of $\mathcal{F}$ by

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} \sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1}. \tag{3.1}$$

Observe that if our family $\mathcal{F}$ contains only elements of size $\lfloor n/2 \rfloor$ we have equality. To prove our result we will further analyze this sum. We start by introducing some notation. Let 1_A be the indicator variable for any event A. Let $\mathcal{C}$ be the set of full chains in $\mathcal{B}_n$. For a diamond-free family $\mathcal{F}$, let $\mathcal{F}_{\min} := \{F \in \mathcal{F} : \nexists\, F' \in \mathcal{F}, F' \subset F\}$ be the set of minimal elements of $\mathcal{F}$. For any $\mathcal{F}' \in \mathcal{F}_{\min}$ let $\mathcal{C}_{F'}$ denote the full chains that contain F' and let $\mathcal{C}_o$ denote the full chains that do not contain a member of $\mathcal{F}_{min}$.

Since the number of chains containing an element $F \in \mathcal{F}$ is $|F|!(n-|F|)!$ we have that

$$\begin{aligned}\sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1} &= \frac{1}{n!}\sum_{F\in\mathcal{F}}\sum_{C\in\mathcal{C}} 1_{F\in C} \\ &= \frac{1}{n!}\sum_{C\in\mathcal{C}_o}\sum_{F\in\mathcal{F}} 1_{F\in C} + \frac{1}{n!}\sum_{F'\in\mathcal{F}_{\min}}\sum_{C\in\mathcal{C}_{F'}}\sum_{F\in\mathcal{F}} 1_{F\in C}\end{aligned}$$

We will now partition $\mathcal{F}$ into three antichains. Let $\mathcal{F}_{\max} = \{F \in (\mathcal{F} - \mathcal{F}_{\min}) : \nexists\, F' \in \mathcal{F}, F \subset F'\}$ and $\mathcal{F}_{\text{mid}} = \mathcal{F} - (\mathcal{F}_{\max} \cup \mathcal{F}_{\min})$. First observe that e for any $F, F' \in \mathcal{F}_{\min}, F \neq F'$ we have that $F \nsubseteq F'$ and $F' \nsubseteq F$ by definition. Hence $\mathcal{F}_{\min}$ is an antichain. Similarly $\mathcal{F}_{\max}$ is antichain. If $\mathcal{F}_{\text{mid}}$ is not an antichain then there exists $A, B \in \mathcal{F}_{\text{mid}}$ such that $A \subset B$. By construction $B \notin \mathcal{F}_{\max}$ and hence there exists $F \in \mathcal{F}_{\max}$ such that $B \subset F$. Also since $A \notin \mathcal{F}_{\min}$ there exists an element $F' \in \mathcal{F}_{\min}$ such that $F' \subset A$. But then $F' \subset A \subset B \subset F$ which forms a 4-chain, which is forbidden by the fact that a diamond is forbidden.

This then implies that any chain $C \in \mathcal{C}_o$ contains at most two elements of $\mathcal{F}$. Therefore $\sum_{F\in\mathcal{F}} 1_{F\in\mathcal{C}_o} \leq 2$. Hence

$$\sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1} \leq \frac{2}{n!}|\mathcal{C}_o| + \frac{1}{n!}\sum_{F'\in\mathcal{F}_{\min}}\sum_{C\in\mathcal{C}_{F'}}\sum_{F\in\mathcal{F}} 1_{F\in C}.$$

For each $F' \in \mathcal{F}_{\min}$ partition $\mathcal{C}_{F'}$ into sets containing all extensions of a chain from F' to $[n]$. Observe that $|\mathcal{C}_{F'}| = |F'|!$. Hence to bound the triple sum we can focus strictly only the intervals $[F', [n]] := \{S : F' \subseteq S \subseteq [n]\}$. The full chains in this interval are denoted $\mathcal{C}'_{F'}$. Therefore, $\sum_{c\in\mathcal{C}_{F'}}\sum_{F\in\mathcal{F}} 1_{F\in\mathcal{C}} = |F'|!\sum_{C\in\mathcal{C}'_{F'}}\sum_{F\in\mathcal{F}} 1_{F\in\mathcal{C}'_{F'}}$ and hence,

$$\begin{aligned}\sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1} &\leq \frac{2}{n!}|\mathcal{C}_o| + \frac{1}{n!}\sum_{F'\in\mathcal{F}_{\min}} \frac{(n-|F|')!|F'|!}{(n-|F'|)!}\sum_{C\in\mathcal{C}'_{F'}}\sum_{F\in\mathcal{F}} 1_{F\in C} \\ &= \frac{2}{n!}|\mathcal{C}_o| + \frac{1}{n!}\sum_{F'\in\mathcal{F}_{\min}} \binom{n}{|F'|}^{-1}\frac{1}{(n-|F'|)!}\sum_{C\in\mathcal{C}'_{F'}}\sum_{F\in\mathcal{F}} 1_{F\in C} \\ &\leq \frac{2}{n!}|\mathcal{C}_o| + \frac{1}{n!}\sum_{F'\in\mathcal{F}_{\min}} \binom{n}{|F'|}^{-1}\Lambda^*(n-|F'|, \mathcal{B}_2).\end{aligned}$$

This is because F' may be regarded as the empty set in the interval $[F', [n]]$. Since $F_{\min}$ is an antichain, it obeys the YBLM inequality (11). Setting $M_n = \max_k\{\Lambda^*(n-k, \mathcal{B}_2) : |k - n/2| < n^{2/3}\}$ and observing that $M_n \geq 2$ for $k \geq 2$ gives

$$\begin{aligned}\sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1} &\leq \frac{2}{n!}|\mathcal{C}_o| + \sum_{F' \in \mathcal{F}_{\min}} \binom{n}{|F'|}^{-1} M_n \\ &\leq \frac{2}{n!}|\mathcal{C}_o| + \left(1 - \frac{|\mathcal{C}_o|}{n!}\right) M_n \\ &\leq M_n.\end{aligned}$$

Now we draw upon some conclusions first proven by Axenovich, Manske and Martin [AMM11] which states $\sum_{|k-n/2| \geq n^{2/3}} \binom{n}{k} \leq 2^{-\Omega(n^{1/3})}\binom{n}{\lfloor n/2 \rfloor}$. We include a proof below for completeness.

Claim 33

$$\sum_{|k-n/2| \geq n^{2/3}} \binom{n}{k} \leq 2^{-\Omega(n^{1/3})}\binom{n}{\lfloor n/2 \rfloor}$$

Proof of Claim 33 Observe that $2^{-n}\sum_{F \in \mathcal{B}_n} 1_{|F|=k,\ |k-n/2| \geq n^{2/3}}$ is the probability of getting a choosing a set from $\mathcal{B}_n$ at random under the uniform distribution. Next we observe that

$$\begin{aligned}2^{-n}\sum_{F \in \mathcal{B}_n} 1_{|F|=k,\ |k-n/2| \geq n^{2/3}} &= 2^{-n}\sum_{F \in \mathcal{B}_n}\sum |k - n/2| \geq n^{2/3} 1_{|F|=k} \\ &= 2^{-n}\sum_{|k-n/2| \geq n^{2/3}} \binom{n}{k}\end{aligned}$$

is the probability that a $B(n, 1/2)$ binomial random variable, X, takes on values outside the interval $(n/2 - n^{2/3}, n/2 + n^{2/3})$. Then a standard Chernoff bound, $\Pr(|X - n/2| \geq \delta(n/2)) \leq 2\exp\{-(n/2)\delta^2/2\}$. Hence, by setting $\delta = 2n^{-1/3}$, we have

$$\sum_{|k-n/2| \geq n^{2/3}} \binom{n}{k} \leq 2^{n+1}e^{-n^{1/3}}.$$

Since $\binom{n}{\lfloor n/2 \rfloor} = \Omega(n^{-1/2})2^n$, we may conclude the result. $\square$

Returning to the proof of Lemma 32, we see that Claim 33 implies that we may assume that $|F'| \in (n/2 - n^{2/3}, n/2 + n^{2/3})$. Then for $k \in (n/2 - n^{2/3}, n/2 + n^{2/3})$ we have that $n - k$

approaches infinity as n approaches infinity and hence

$$\sum_{F\in\mathcal{F}} \binom{n}{|F|}^{-1} \le M_n = \max_k \left\{ \Lambda^*(n-k,\mathcal{B}_2) : |k-n/2| < n^{2/3} \right\}$$
$$\le \Lambda^*(\mathcal{B}_2) + o(1).$$

Combining this with inequality (3.1) we have that $|\mathcal{F}| \le \left(\Lambda^*(\mathcal{B}_2)+o(1)\right)\binom{n}{\lfloor n/2\rfloor}$. This concludes the proof of Lemma 32. □

3.2 Converting from a Poset Upper Bound to a Graph Parameter Upper Bound

Definition 34 *For a graph G, let $\alpha_i = \alpha_i(G)$ denote the number of three-vertex subgraphs that induce exactly i edges for $i = 0,1,2,3$ and let $\beta_j = \beta_j(G)$ denote the number of four-vertex subgraphs that induce exactly j edges for $j = 0,\ldots,6$. If (X,Y) is an ordered bipartition of $V(G)$, then let $\overline{e}(X)$ denote the number of nonedges in the subgraph induced by X and $\overline{e}(Y)$ denote the number of nonedges in the subgraph induced by Y.*

For every nonnegative integer k and real number n, we use the Pochhammer symbol $(n)_k = n(n-1)\cdots(n-k+1)$ for $k\ge 1$ and $(n)_0 = 1$.

Lemma 35 *For every $\mathcal{B}_2$-free family $\mathcal{F}$ in $\mathcal{B}_n$ with $\emptyset\in\mathcal{F}$, there exist the following:*

- *a graph $G=(V,E)$ on $v\le n$ vertices and*
- *a set $W=\{w_{v+1},\ldots,w_n\}$ such that, for each $w\in W$, (X_w,Y_w) is an ordered bipartition of V;*

for which

$$\Lambda(n,\mathcal{F}) \le 2 + f(n,G,W),$$

where, with the notation as above,

$$f(n,G,W) = \frac{2\alpha_1(G)-2\alpha_2(G)}{(n)_3} + \frac{6\beta_0(G)}{(n)_4} + \sum_{w\in W}\left[\frac{|X_w|-|Y_w|}{(n)_2} + \frac{4\overline{e}(Y_w)-2\overline{e}(X_w)}{(n)_3}\right]. \quad (3.2)$$

Proof. We begin by partitioning the set of full chains, $\mathcal{C}$ in $\mathcal{B}_n$ in the following manner. For a family of sets $\mathcal{F} \subseteq \mathcal{B}_n$, denote $\Psi_i = \Psi_i(\mathcal{F})$ to be the set of full chains that contain exactly i members of $\mathcal{F}$ for $i = 0, 1, \ldots, n+1$. Note that $|\Psi_0| = 0$ since $\emptyset \in \mathcal{F}$ by hypothesis and $|\Psi_i(\mathcal{F})| = 0$ for $i \geq 4$ since $\mathcal{F}$ cannot contain a chain of length 4. Hence $\Psi_1 \cup \Psi_2 \cup \Psi_3 = \mathcal{C}$.

Since every chain can only contain 1, 2 or 3 elements in any $\mathcal{B}_2$-free poset $\mathcal{F}$ we have that $n! = |\Psi_1| + |\Psi_2| + |\Psi_3|$. Then for each $i \in \{1, 2, 3\}$ the total number of times all the chains from Ψ_i contains an element of $\mathcal{F}$ is $i|\Psi_i|$. Recall that $\Lambda(n, \mathcal{F})$ is the average number of times a full chain in $\mathcal{B}_n$ contains an element of $\mathcal{F}$. Hence we have that $\Lambda(n, \mathcal{F}) = \frac{3|\Psi_3|+2|\Psi_2|+|\Psi_1|}{n!}$. Therefore $\Lambda(n, \mathcal{F}) = 2 + \frac{|\Psi_3|-|\Psi_1|}{n!}$.

Let $W = \{w \in [n] : \{w\} \in \mathcal{F}\}$ and $V = [n] - W$. Then W is the set of all singletons in $\mathcal{F}$, and V is the set of all singletons not in $\mathcal{F}$. Let $v = |V|$. Without loss of generality, let $W = \{w_{v+1}, w_{v+2}, \ldots, w_n\}$. Define the graph $G = (V, E(G))$ where

$$E(G) = \{\{v, v'\} : v, v' \in V \text{ and } \{v, v'\} \in \mathcal{F}\}.$$

For every $w \in W$, we create an ordered bipartition of V called (X_w, Y_w) where

$$X_w = \{x \in V : \{x, w\} \in \mathcal{F}\} \quad \text{and} \quad Y_w = V - X_w.$$

Note that for all $w, w' \in W, w \neq w'$, $w' \notin X_w$ else $\{\emptyset, \{w\}, \{w'\}, \{w, w'\}\}$ would form a diamond in $\mathcal{F}$.

We proceed by placing bounds on $|\Psi_1|$ and $|\Psi_3|$. First, we find a lower bound on $|\Psi_1|$. To do so, we must have a list that counts chains contained in Ψ_1 such that no chain is counted twice or that any such double count is corrected. The following is a list of chains contained in Ψ_1 along with the number of such chains.

(a) Let $z_1, z_2, z' \in V$ be a set of 3 vertices that induces exactly two edges with $z_1 z_2 \notin E(G)$ Then Ψ_1 includes the chains of the form $\emptyset, \{z_1\}, \{z_1, z_2\}, \{z_1, z_2, z'\}, \ldots, [n]$ and chains of the form $\emptyset, \{z_2\}, \{z_2, z_1\}, \{z_2, z_1, z'\}, \ldots, [n]$. Let the set of these chains be denoted by (a). Then each $z \in V$ is counted $(n-3)!$ times there is a labeled 3 vertex subgraph of G containing v that induces exactly two edges. Hence there are $2\alpha_2 \cdot (n-3)!$ such chains.

(b) Let $w \in W$, $y \in Y_w$ and $z' \in V - \{y\}$ such that $yz' \in E(G)$. Then Ψ_1 includes the chains of the form $\emptyset, \{y\}, \{y, w\}, \{y, w, z'\}, \ldots, [n]$. Let the set of these chains be denoted by (b). Then each $w \in W$ is counted $(n-3)!$ times for every edge incident with $y \in Y_w$. Hence there are $\sum_{w \in W} \sum y \in Y_w \, [\deg(y) \cdot (n-3)!]$ such chains, where $\deg(y)$ is the degree of the vertex y in the graph induced by V.

(c) Let $w \in W$, $x \in X_w$ and $z' \in V - \{x\}$ such that $xz' \notin E(G)$. Then Ψ_1 includes the chains of the form $\emptyset, \{x\}, \{x, z'\}, \{x, z', w\}, \ldots, [n]$. Let the set of these chains be denoted by (c). Then each $w \in W$ is counted $(n-3)!$ times each time there exists a $z \in V - \{x\}$ such that $xz' \notin E(G)$. Hence there are $\sum_{w \in W} \sum_{x \in X_w} \left[\overline{\deg}(x) \cdot (n-3)!\right]$ such chains, where $\overline{\deg}(x)$ is the degree of x in the complement of the graph induced by V. This can also be considered the nondegree of x or equivalently $\overline{\deg}(x) = v - 1 - \deg(x)$.

(d) Let $w \in W$, $w' \in W - \{w\}$ and $y \in Y_w$. Then Ψ_1 includes chains of the form $\emptyset, \{y\}, \{y, w\}, \{y, w, w'\}, \ldots, [n]$. Let the set of these chains be denoted by (d). Since each $w \in W$ is counted $(n-3)!$ times for each $y \in Y_W$, there are $\sum_{w \in W} [|Y_w|(|W| - 1) \cdot (n-3)!]$ such chains.

We now prove that a chain is not double counted in these sets of chains . The proof is somewhat clear and rudimentary but is included for completeness.

Claim 36 *A full chain in $\mathcal{B}_n$ can cannot be contained in two distinct sets (a), (b), (c) or (d).*

Proof of Claim 36 We proceed by showing that the intersection of these sets is empty.

Suppose that a full chain C in $\mathcal{B}_n$ is of the form found in (a). Then the singleton and the doubleton elements in the chain consist of elements from V. Since $V \cap W = \emptyset$, C cannot have the form found in (b), (c) or (d).

Suppose that a full chain C in $\mathcal{B}_n$ is of the form found in (b). Since the singleton $\{w\}$ found in C is a subset of W, C cannot be of the form found in the sets (a) or (d). Let the doubleton in C be $\{w, y\}$ where $w \in W$ and $y \in Y_w$. Since $X_w \cap Y_w = \emptyset$, C is not of the form found in the set (c).

Suppose that a full chain C in $\mathcal{B}_n$ is of the form found in (c). Since the singleton $\{w\}$ found in C is a subset of W, C cannot be of the form found in the sets (a) or (d). Let the doubleton in C be $\{w, x\}$ where $w \in W$ and $x \in X_w$. Since $X_w \cap Y_w = \emptyset$, C is not of the form found in the set (b).

Suppose that a full chain C in $\mathcal{B}_n$ is of the form found in (d). Let the doubleton in C be $\{y, w\}$ where $w \in W$ and $y \in Y_w$. Since the singleton in C is $\{y\}$, C cannot be of the form found in the sets (a) or (c). Since the element of order 3 in C contains two elements of W, C cannot be of the form (b).

Therefore the each of the pairwise intersections of (a), (b), (c) and (d) is empty. This completes the proof of Claim 36. $\square$

Returning to the proof of Lemma 35, we observe that it is not necessarily true that all of the chains found in Ψ_1 are contained in one of the sets (a), (b), (c) or (d), but since each chain is only counted once over the union of these sets we have that the size of Ψ_1 is lower bounded by the sum of the order of (a), (b), (c) and (d). Dividing this sum by $(n-3)!$ gives us

$$\frac{|\Psi_1|}{(n-3)!} \geq 2\alpha_2 + \sum_{w \in W} \sum_{y \in Y_w} \deg(y) + \sum_{w \in W} \sum_{x \in X_w} \overline{\deg}(x) + \sum_{w \in W} |Y_w|(|W|-1).$$

Next we take a closer look at $\sum_{y \in Y_w} \deg(y) + \sum_{x \in X_w} \overline{\deg}(x)$. Fix $w \in W$. Observe that if there exists an edge between two vertices in Y_w then the first sum will count that edge twice. Also if no edge exists between two vertices in Y_w then neither sum will increase in count. If there exist two vertices in X_w without an edge between them then the second sum will count that non-edge twice. Also if there exists an edge between two vertices in X_w then neither sum will count that edge. All that remains to analyze is if an edge exists or not between a vertex in X_w and a vertex in Y_w. Consider an edge between vertices in X_w and Y_w. The first sum will count the incidence with the vertex in Y_w but the second sum will not. Suppose there isn't an edge between vertices in X_w and Y_w. Then the second sum will count this non-incidence for the vertex in X_w but the first sum will not. Hence if we have a vertex in Y_w and a vertex in X_w no matter if an edge exists or not the pair of sums will count this occurrence once. Let $e(Y_w) = |E(Y_w)|$ and $\overline{e}(X_w) = |E(X_w)|$. Therefore $\sum_{y \in Y_w} \deg(y) + \sum_{x \in X_w} \overline{\deg}(x) = |X_w||Y_w| + 2e(Y_w) + 2\overline{e}(X_w)$

and

$$\frac{|\Psi_1|}{(n-3)!} \geq 2\alpha_2 + \sum_{w \in W} \left[|X_w||Y_w| + 2e(Y_w) + 2\overline{e}(X_w) + (|W|-1)|Y_2| \right]. \tag{3.3}$$

Next consider $|\Psi_3|$. Since we must establish an upper bound for $|\Psi_3|$ it is necessary to count every chain containing 3 elements of $\mathcal{F}$. It is important to separate the chains in Ψ_3 into chains that have a singleton $\{w\}$ in $\mathcal{F}$ (that is, chains of the form $\emptyset, \{w\}, \ldots, [n]$ for some $w \in W$) and those that do not contains a singleton found in $\mathcal{F}$. Let $\mathcal{T}$ be the minimal elements of $\mathcal{F} - \{\emptyset\}$ and $\mathcal{U}$ be $\mathcal{F} - \{\emptyset\} - \mathcal{T}$. The following is a list of all the chains in Ψ_3 (to be proven in Claim 37):

(i) Let $w \in W$ and $x \in X_w$. Then Ψ_3 contains chains of the form $\emptyset, \{w\}, \{x, w\}, \ldots, [n]$. Each $w \in W$ is counted $(n-3)!$ times for each $x \in X_w$. Hence there are $\sum_{w \in W} |X_w| \cdot (n-3)!$ such chains of this type. Let the set of these chains be called (i).

(ii) Let $w \in W$, $U \in \mathcal{U}$ and $|U| \geq 3$ such that $w \in U$. Then Ψ_3 contains chains of the form $\emptyset, \{w\}, \ldots, U, \ldots, [n]$. There are at most $\sum_{w \in W} \left(|Y_w|^2 - |Y_w| - 2e(Y_w)\right) \cdot (n-3)!$ of these types of chains. This will be proven in Claim 38. Let the set of these chains be called (ii).

(iii) Let $v_1, v_2, v' \in V$ be a set of 3 vertices that induces exactly one edge with $v_1v_2 \in E(G)$. Then Ψ_3 contains chains of the form $\emptyset, \{v_1\}, \{v_1, v_2\}, \{v_1, v_2, v'\}, \ldots, [n]$ and chains of the form $\emptyset, \{v_2\}, \{v_2, v_1\}, \{v_2, v_1, v'\}, \ldots, [n]$. Each $v \in V$ gets counted $2(n-3)!$ times and there is a labeled 3 vertex subgraph of G that induces one edge incident to v. Hence the number of chains of this type is $2\alpha_1 \cdot (n-3)!$. Let the set of these chains be called (iii).

(iv) Let $U \subset V$ be a member of $\mathcal{U}$ with $|U| \geq 4$ where U does not contain an edge in G between any two elements $u, v \in U$. Then Ψ_3 contains chains of the form $\emptyset, \ldots, U, \ldots, [n]$. Note that only one additional member of $\mathcal{F}$ can be in the interval $[\emptyset, u]$. Hence at most a $1/|U|$ fraction of these chains contain three members of $|F|$. There are at most $\frac{6}{n-3}\mathcal{B}_o \cdot (n-3)!$ such chains. This bound will be proven in Claim 39. Let the set of these chains be called (iv).

In order to insure that we have an upper bound for $|\Psi_3|$ we must verify that we contain all chains in $|\Psi_3|$.

Claim 37 *The chains found in (i), (ii), (iii) and (iv) is an exhaustive list of the chains found in Ψ_3.*

Proof of Claim 37 Let C be a full chain in $\mathcal{B}_n$ containing three elements of $\mathcal{F}$. Note that every chain must contain $\emptyset$ and recall that $\mathcal{T}$ is the minimal elements of $\mathcal{F} - \{\emptyset\}$. Since C contains 3 elements of $\mathcal{F}$, there must be a $T \in C$ such that $T \in \mathcal{T}$. Let U be the third element in C and observe that $|U| > |T|$. Suppose $|T| = 2$. Then $T = \{w\}$ for some $w \in W$. (Recall that W contains all the singletons of $\mathcal{F}$.) Let $|U| = 2$. Then C is of the form found in (i). Suppose $|U| \geq 3$. Then C is of the form found in (ii). Suppose now that $|T| = 2$. Then T induces an edge in G and we have a chain in the set (iii). If $|T| \geq 3$ then the chain is of the form found in (iv). Therefore, we have counted all possible chains that contain 3 elements of $\mathcal{F}$. This completes the proof of Claim 37. $\square$

Claim 38 *For a fixed $w \in W$ the number of chains of type (ii) is bounded above by*

$$\big(|Y_w|^2 - |Y_w| - 2e(Y_w)\big) \cdot (n-3)!.$$

Hence, the total number of such chains is at most

$$\sum_{w \in W} \big(|Y_w|^2 - |Y_w| - 2e(Y_w)\big) \cdot (n-3)!.$$

Proof of Claim 38. Fix $w \in W$. Then for any $U \in \mathcal{U}$ such that $w \in U$ there are $|w|!(|U| - |w|)!(n-|U|)! = (|U|-1)!(n-|U|)!$ chains that pass through w and U. Hence we may bound the number of chains of type (ii) as follows:

$$\begin{aligned}
&\sum_{U \in \mathcal{U}: U \ni w, |U| \geq 3} (|U|-1)!(n-|U|)! \\
&= (n-3)! \sum_{U \in \mathcal{U}: U \ni w, |U| \geq 3} \binom{|Y_w|}{|U|-1}^{-1} \frac{(|Y_w|)_{|U|-1}}{(n-3)_{|U|-3}} \\
&= (n-3)!|Y_w|(|Y_w|-1) \sum_{U \in \mathcal{U}: U \ni w, |U| \geq 3} \binom{|Y_w|}{|U|-1}^{-1} \frac{(|Y_w|-2)_{|U|-3}}{(n-3)_{|U|-3}}
\end{aligned}$$

Since $|Y_2| \leq n-1$, we have that $\frac{(|Y_w|-2)_{|U|-3}}{(n-3)_{|U|-3}} \leq 1$. Furthermore, consider subsets of Y_w. Suppose there exists an $x \in (U-\{w\}) \cup X_w$. By definition $x, w \in \mathcal{F}$. But then $\emptyset \subset \{w\}, \{x,w\}, \subset U$ forms a diamond in $\mathcal{F}$. Hence $(U-\{w\}) \subseteq Y_w$. Now observe that no edge y_1y_2 can be a subset of the set $U-\{w\}$ for any U, else $\emptyset \subset \{w\}, \{y_1, y_2\} \subset U$ would form a diamond. Also, if two sets $U_1, U_2 \in \mathcal{U}$, such that $U_1 \subset U_2$ then $\emptyset \subset \{w\}, U_1 \subset U_2$ would form a diamond. Therefore $\{U-\{w\} : U \in \mathcal{U}, U \ni w, |U| \geq 3\} \cup E\left(G|_{Y_w}\right)$ forms an antichain in $\mathcal{B}_{|Y_w|}$ and hence the YBLM inequality gives $e(Y_w)\binom{|Y_w|}{2}^{-1} + \sum_{U\in\mathcal{U}:U\ni w,|U|\geq 3}\binom{|Y_w|}{|U|-1}^{-1} \leq 1$. Therefore

$$\begin{aligned}
&\sum_{U\in\mathcal{U}:U\ni w,|U|\geq 3} (|U|-1)!(n-|U|)! \\
&\leq (n-3)!|Y_w|(|Y_w|-1) \sum_{U\in\mathcal{U}:U\ni w,|U|\geq 3} \binom{|Y_w|}{|U|-1}^{-1} \\
&\leq (n-3)!|Y_w|(|Y_w|-1)\left(1-e(Y_w)\binom{|Y_w|}{2}^{-1}\right) \\
&= (n-2)!\left(|Y_w|^2 - |Y_w| - 2e(Y_w)\right).
\end{aligned}$$

This concludes the proof of Claim 38.

□

Claim 39 *The number of chains of type (iv) is at most* $\frac{6}{n-3}\beta_0 \cdot (n-3)!$.

Proof of Claim 39.

We start with an approach that is discussed in Axenovich, Mankse and Martin [AMM11]. A set U *covers* a set T if there exists some $u \in U$ such that $T = U - \{u\}$. Fix some $T \in \mathcal{T}$ with $|T| \geq 3$, then we will count these chains by replacing the members of $\mathcal{U}$ that are supersets of $T \in \mathcal{T}$ with superset of T that covers T.

Let $\mathcal{U}(T)$ denote the members of $\mathcal{U}$ that are supersets of T and $\mathcal{U}_0(T)$ denote the family of all sets U such that U covers T but there is no other member of $\mathcal{T}$ that U covers. Note that members of $\mathcal{U}_0(T)$ need not be members of $\mathcal{U}$. Note further that, by definition, each $\mathcal{U}_0(T)$ is an antichain.

Consider the family $\mathcal{U}' = \bigcup_{T\in\mathcal{T}} \mathcal{U}_0(T)$. If there exists $U \in \mathcal{U}_0(T) \cap \mathcal{U}$ then for all $T' \in \mathcal{T}$, where $T' \neq T$, $U \notin \mathcal{U}_0(T')$ else $\emptyset \subset T, T' \subset U$ will form a diamond in $\mathcal{F}$. Also if $U_1, U_2 \in \mathcal{U}'$

such that $U_1 \subset U_2$ then there are distinct $T_1, T_2 \in \mathcal{T}$ such that $T_1 \subset U_1$ and $T_2 \subset U_2$, but then $U_2 \supset T_1$ contradicting the definition of $\mathcal{U}_0(T_2)$. Since $\mathcal{T}$ is an antichain by definition, $\{\emptyset\} \cup \mathcal{T} \cup \mathcal{U}'$ does not contain a diamond.

Furthermore, any chain that contains T and some $U \in \mathcal{U}(T)$ has the property that there is some $U_0 \in \mathcal{U}_0(T)$ such that this chain also contains U_0. Suppose not, then the chain contains some X that covers T for which $T \subset X \subset U$. If $X \notin \mathcal{U}_0(T)$ then there is a $T' \in \mathcal{T}$, $T \neq T'$ such that $X \supset T'$. But then $\emptyset \subset T, T' \subset U$ is a diamond in $\mathcal{F}$.

Recall that for each $|U|$ at most $1/|U|$ fraction of the chains of type (iv) contain 3 members of $\mathcal{F}$. Hence we can bound the number of chains of type (iv) as follows:

$$\begin{aligned}
|(iv)| &\leq \sum_{U\in\mathcal{U}, U\subseteq V, |U|\geq 4} \frac{1}{|U|}(|U|)!(n-|U|)! \\
&= (n-3)! \sum_{U\in\mathcal{U}, U\subseteq V, |U|\geq 4} \frac{1}{|U|} \cdot \frac{|U|!}{(n-3)_{|U|-3}} \\
&= (n-3)! \frac{(v)_4}{(n-3)} \sum_{U\in\mathcal{U}, U\subseteq V, |U|\geq 4} \frac{1}{|U|} \binom{v}{|U|}^{-1} \frac{(v-4)_{|U|-4}}{(n-4)_{|U|-4}}.
\end{aligned}$$

Since $v \leq n$, $\frac{(v-4)_{|U|-4}}{(n-4)_{|U|-4}} \leq 1$. Furthermore $1/|U| \leq 1/4$ for all U over the sum. Therefore,

$$|(iv)| \leq (n-3)! \frac{(v)_4}{(n-3)} \sum_{U\in\mathcal{U}, U\subseteq V, |U|\geq 4} \frac{1}{4} \binom{v}{|U|}^{-1}.$$

Observe that for all $U \in \{U \in \mathcal{U} : U \subseteq V, |U| \geq 4\}$ there does not exist $x, y \in U$ such that $xy \in E(G)$ otherwise $\emptyset \subseteq \{x, y\}, T \subseteq U$ is a diamond in $\mathcal{F}$. Hence $\{U \in \mathcal{U} : U \subseteq V, |U| \geq 4\}$ together with subsets of V of size four that induce at least one edge will form an antichain. By the YBLM inequality that gives $(\binom{v}{4} - \beta_0)\binom{v}{4}^{-1} + \sum \binom{v}{|U|}^{-1} \leq 1$. Note that the set over which we sum is suppressed for the ease of reading but we are summing over $U \in \{U \in \mathcal{U} : U \subseteq V, |U| \geq 4\}$. Simplifying this gives $\sum \binom{v}{|U|}^{-1} \leq \beta_0 \binom{v}{4}^{-1}$. Hence,

$$\begin{aligned}
|(iv)| &\leq (n-3)! \frac{(v)_4}{(n-3)} \sum_{U\in\mathcal{U}: U\subseteq V, |U|\geq 4} \frac{1}{4} \binom{v}{|U|}^{-1} \\
&\leq (n-3)! \frac{(v)_4}{4(n-3)} \beta_0 \binom{v}{4}^{-1} \\
&= (n-3)! \frac{6}{n-3} \beta_0.
\end{aligned}$$

This completes the proof of Claim 39.

□

Now we return to the proof of Lemma 35. By combining the bounds on Ψ_1 and Ψ_3 we have that

$$\begin{aligned}
\frac{|\Psi_3| - |\Psi_1|}{(n-3)!} \leq 2\alpha_1 + \frac{6}{n-3}\beta_0 + \sum_{w \in W} [|X_w|(n-2) + |Y_w|(|Y_w| - 1) - 2e(Y_w)] \\
- 2\alpha_2 - \sum_{w \in W} [|X_w||Y_w| + 2e(Y_w) + 2\overline{e}(X_w) + (|W| - 1)|Y_w|] \\
= 2(\alpha_1 - \alpha_2) + \frac{6}{n-3}\beta_0 \\
+ \sum_{w \in W} [|X_w|(n-2) + (|Y_w|)_2 - |Y_w|(|W| - 1) - |X_w||Y_w| \\
-4e(Y_w) - 2\overline{e}(X_w)] \\
= 2(\alpha_1 - \alpha_2) + \frac{6}{n-3}\beta_0 \\
+ \sum_{w \in W} [|X_w|(n-2) + |Y_w|(|Y_w| - |W|) - |X_w||Y_w| \\
-2(|Y_w|)_2 + 4\overline{e}(Y_w) - 2\overline{e}(X_w)]
\end{aligned}$$

Since V and W partition the singletons in $\mathcal{B}_n$ and for any $w \in W$, X_w and Y_w partition V we have that $|W| + |Y_w| + |X_w| = n$ regardless of w. Then rearranging the values in the sum and observing that $|Y_w|(|Y_w| - |W| - |X_w| - 2|Y_w| + 2) = -|Y_w|(n-2)$ we can simplify the last expression as follows:

$$\begin{aligned}
\frac{|\Psi_3| - |\Psi_1|}{(n-3)!} \leq 2(\alpha_1 - \alpha_2) + \frac{6}{n-3}\beta_0 \\
+ \sum_{w \in W} [(|X_w| - |Y_w|)(n-2) + 4\overline{e}(Y_w) - 2\overline{e}(X_w)].
\end{aligned}$$

Dividing by $(n)_3$ gives us the desired bound on $3|\Psi_3| + 2|\Psi_2| + |\Psi_1| = 2 + |\Psi_3| - |\Psi_1|$. This concludes the proof of Lemma 35.

□

3.3 Placing an Upper Bound on the Graph Parameter $f(n, G, W)$

Lemma 40 *For any integer n, graph $G = (V, E)$ on $v \le n$ vertices and a set W, of $n - v$ bipartitions of $V(G)$,*

$$f(n, G, W) \le \frac{1}{4} + O\left(\frac{1}{n}\right)$$

where $f(n, G, W)$ is as defined in Lemma 35.

Proof.

We will proceed by calculating $f(n, G, W)$ by summation on the set of induced subgraphs on 4 vertices. We first eliminate the case when $v \le 3$.

If $v \le 3$ then $\alpha_1 \le 1$, $|X_w| \le 3$ and $\overline{e}(Y_w) \le 3$. It then trivially follows that

$$f(n, G, W) \le \frac{2}{(n)_3} + n\left(\frac{3}{(n)_2} + \frac{12}{(n)_3}\right),$$

which is at most $1/4$ if $n \ge 11$.

Suppose $v \ge 4$. Let $\mathcal{H}_4 = \binom{V}{4}$, the set of all 4-tuples of vertices. So as to not overburden the reader with notation if $H \in \mathcal{H}_4$ we will use H to mean both the set of 4 vertices and the subgraph induced by said vertices.

Now we make some basic observations. First, $\sum_{H \in \mathcal{H}_4} \alpha_i(H) = (v-3)\alpha_i(G)$ for $i = 0, 1, 2, 3$ since every triplet will get counted once for every other vertex in G. Second, for any set $S \subseteq V$, we have that $\sum_{H \in \mathcal{H}_4} \overline{e}(S \cap H) = \binom{v}{2}\overline{e}(S)$ since every pair of vertices without an edge in S will get counted once for ever other distinct pair of vertices. Third, $\sum_{H \in \mathcal{H}_4} |S \cap H| = \binom{v}{3}|S|$ since every vertex will get counted once for every other triple in G. Hence we may rewrite $f(n, G, W)$ as follows:

$$\begin{aligned}
f(n, G, W) &= \frac{2\alpha_1 - 2\alpha_2}{(n)_3} + \frac{6\beta_0}{(n)_4} + \sum_{w \in W}\left[\frac{|X_w| - |Y_w|}{(n)_2} + \frac{4\overline{e}(Y_w) - 2\overline{e}(X_w)}{(n)_3}\right] \\
&= \frac{1}{\binom{v}{4}} \sum_{H \in \mathcal{H}_4}\Bigg[\frac{(v)_3}{3(n)_3} \cdot \frac{\alpha_1(H) - \alpha_2(H)}{4} + \frac{(v)_4}{4(n)_4}\beta_0(H) \\
&\qquad + \sum_{w \in W}\Bigg(\frac{v}{(n)_2} \cdot \frac{|X_w \cap H| - |Y_w \cap H|}{4} \\
&\qquad\qquad + \frac{(v)_2}{2(n)_3} \cdot \frac{4\overline{e}(Y_w \cap H) - 2\overline{e}(X_w \cap H)}{6}\Bigg)\Bigg].
\end{aligned}$$

Our goal is to minimize this sum as much as possible. In Claim 42 we will see that for most cases of $H \in \mathcal{H}_4$, the largest values for this sum are when $X_w \cap H = H$. We will then rearrange the terms so that the most relevant parts are prominent. The rearrangement is as follows:

$$f(n,G,W) = \frac{1}{\binom{v}{4}} \sum_{H \in \mathcal{H}_4} \left[\frac{(v)_3}{3(n)_3} \cdot \frac{\alpha_1(H) - \alpha_2(H)}{4} + \frac{(v)_4}{4(n)_4}\beta_0(H) + \frac{(n-v)v}{(n)_2} \right.$$
$$\left. + \frac{(n-v)(v)_2\,\overline{e}(H)}{6(n)_3} + \sum_{w \in W} \epsilon(n,w,G,H) \right], \tag{3.4}$$

where

$$\epsilon(n,w,G,H) = -\frac{v}{(n)_2} \cdot \frac{2|Y_w \cap H|}{4} + \frac{(v)_2}{2(n)_3} \cdot \frac{2\overline{e}(H) + 4\overline{e}(Y_w \cap H) - 2\overline{e}(X_w \cap H)}{6}.$$

Now we bring forth notation presented in Axenovich, Manske and Martin [AMM11] to describe the eleven distinct nonisomorphic graphs on exactly 4 vertices. These will be the same subgraphs that we will use in our flag algebra calculations that will be motivation for the values we add in Case 2.

Definition 41 *For $i = 0,1,5,6$, the 4-vertex graph with exactly i edges is denoted H_i.*

The graph with exactly two edges that are incident is $H_\wedge$ and the graph with exactly two edges that are nonincident is $H_\|$. Their complements are H_Q and $H_\square$, respectively.

The graph inducing a star with three edges (the claw) is $H_\dashv$, the graph inducing a triangle is $H_\triangle$ and the path with three edges is $H_\sqcup$.

We use $\cong$ to denote that two graphs are isomorphic.

We now present a bound for the ϵ term found in equation (3.4). Note that for all but three nonisomorphic subgraphs on 4 vertices we find that ϵ is less than zero.

Claim 42 *For any integer n, graph $G=(V,E)$ on $v<n$ vertices, ordered bipartition (X_w,Y_w) of V and $H\in\mathcal{H}_4$,*

$$\epsilon(n,w,G,H)\leq\begin{cases}\frac{3v}{(n)_2}\max\{0,\frac{v-1}{n-2}-\frac{2}{3}\}, & \text{if } H\cong H_0;\\ \frac{5v}{2(n)_2}\max\{0,\frac{v-1}{n-2}-\frac{4}{5}\}, & \text{if } H\cong H_1;\\ \frac{5v}{3(n)_2}\max\{0,\frac{v-1}{n-2}-\frac{9}{10}\}, & \text{if } H\cong H_\wedge;\\ 0, & \text{otherwise.}\end{cases}$$

Proof of Claim 42.

Note that we have defined $\epsilon(n,w,G,H)$ to be zero if $|Y_w\cap H|=\emptyset$.

We continue by an exhaustive search based on $|Y_w\cap H|$. We will choose the partition of the set of vertices in H that make $4\overline{e}(Y_w\cap H)-2\overline{e}(X_w\cap H)$ as large as possible. First suppose that $\overline{e}(H)\leq 3$. Then $4\overline{e}(Y_w\cap H)-2\overline{e}(X_w\cap H)$ will be maximized when $|Y_w\cap H|\in\{3,4\}$ else there will be a non-edge incident with a vertex in X_w. Note that this case includes the subgraphs $H_6,H_5,H_Q,H_\Box,H_\triangle$ and $H_\sqcup$. In all of the figures for this claim, all of the calculations are basic arithmetic and assuming $\overline{e}(X_w\cap H)$ is zero. Table 3.1 gives the concise values and a bound that shows that $\epsilon(n,2,G,H)\leq 0$ in these cases.

$\lvert Y_w\cap H\rvert$	maximum value of $\epsilon(n,w,G,H)$ if $\overline{e}(H)\leq 3$.			
1	$-\frac{v}{2(n)_2}+\frac{(v)_2}{2(n)_3}\frac{2\overline{e}(H)}{6}$	$\leq$	$\frac{v}{2(n)_2}\left(-1+\frac{v-1}{n-2}\cdot\frac{\overline{e}(H)}{3}\right)$	≤ 0
2	$-\frac{v}{(n)_2}+\frac{(v)_2}{2(n)_3}\frac{2\overline{e}(H)+4\cdot 1}{6}$	$\leq$	$\frac{v}{(n)_2}\left(-1+\frac{v-1}{n-2}\cdot\frac{\overline{e}(H)+2}{6}\right)$	<0
3	$-\frac{3v}{2(n)_2}+\frac{(v)_2}{2(n)_3}\frac{2\overline{e}(H)+4\overline{e}(H)}{6}$	$=$	$\frac{v}{(n)_2}\left(-\frac{3}{2}+\frac{v-1}{n-2}\cdot\frac{\overline{e}(H)}{2}\right)$	≤ 0
4	$-\frac{2v}{(n)_2}+\frac{(v)_2}{2(n)_3}\frac{2\overline{e}(H)+4\overline{e}(H)}{6}$	$=$	$\frac{v}{(n)_2}\left(-2+\frac{v-1}{n-2}\cdot\frac{\overline{e}(H)}{2}\right)$	<0

Table 3.1: Maximum value of $\epsilon(n,w,G,H)$ if $\overline{e}(H)\leq 3$.

We now examine the remaining nonisomorphic four vertex subgraphs individually. Table 3.2 gives the bounds for $H_\parallel$. Note that $\epsilon(n,w,G,H_\parallel)\leq 0$ for all cases.

$\lvert Y_w \cap H\rvert$	maximum value of $\epsilon(n,w,G,H_{\|})$			
1	$-\frac{v}{2(n)_2} + \frac{(v)_2}{2(n)_3}\frac{2\cdot 4-2\cdot 2}{6}$	$=$	$\frac{v}{2(n)_2}\left(-\frac{1}{2}+\frac{1}{3}\frac{v-1}{n-2}\right)$	<0
2	$-\frac{v}{(n)_2} + \frac{(v)_2}{2(n)_3}\frac{2\cdot 4+4\cdot 1-2\cdot 1}{6}$	$=$	$\frac{v}{(n)_2}\left(-1+\frac{5}{6}\frac{v-1}{n-2}\right)$	<0
3	$-\frac{3v}{2(n)_2} + \frac{(v)_2}{2(n)_3}\frac{2\cdot 4+4\cdot 3}{6}$	$=$	$\frac{v}{(n)_2}\left(-\frac{3}{2}+\frac{5}{3}\frac{v-1}{n-2}\right)$	<0
4	$-\frac{2v}{(n)_2} + \frac{(v)_2}{2(n)_3}\frac{6\cdot 4}{6}$	$=$	$\frac{v}{(n)_2}\left(-2+2\frac{v-1}{n-2}\right)$	≤ 0

Table 3.2: Maximum value of $\epsilon(n,w,G,H_{\|})$.

Table 3.3 gives the bounds for $H_\wedge$. Note that $\epsilon(n,w,G,H_\wedge) \leq 0$ unless $|Y_w \cup H| = 3$ and v is large enough.

$\lvert Y_w \cap H\rvert$	maximum value of $\epsilon(n,w,G,H_\wedge)$			
1	$-\frac{v}{2(n)_2} + \frac{(v)_2}{2(n)_3}\frac{2\cdot 4-2\cdot 1}{6}$	$=$	$\frac{v}{2(n)_2}\left(-\frac{1}{2}+\frac{1}{2}\frac{v-1}{n-2}\right)$	≤ 0
2	$-\frac{v}{(n)_2} + \frac{(v)_2}{2(n)_3}\frac{2\cdot 4+4\cdot 1}{6}$	$=$	$\frac{v}{(n)_2}\left(-1+\frac{v-1}{n-2}\right)$	<0
3	$-\frac{3v}{2(n)_2} + \frac{(v)_2}{2(n)_3}\frac{2\cdot 4+4\cdot 3}{6}$	$=$	$\frac{5v}{3(n)_2}\left(-\frac{9}{10}+\frac{v-1}{n-2}\right)$	
4	$-\frac{2v}{(n)_2} + \frac{(v)_2}{2(n)_3}\frac{6\cdot 4}{6}$	$=$	$\frac{v}{(n)_2}\left(-2+2\frac{v-1}{n-2}\right)$	≤ 0

Table 3.3: Maximum value of $\epsilon(n,w,G,H_\wedge)$.

Table 3.4 gives the bounds for H_1. Note that $\epsilon(n,w,G,H_1) \leq 0$ or v is very large in which case the maximum value occurs when $|Y_w \cup H| = 4$.

$\lvert Y_w \cap H\rvert$	maximum value of $\epsilon(n,w,G,H_1)$			
1	$-\frac{v}{2(n)_2} + \frac{(v)_2}{2(n)_3}\frac{2\cdot 5-2\cdot 2}{6}$	$=$	$\frac{v}{2(n)_2}\left(-\frac{1}{2}+\frac{1}{2}\frac{v-1}{n-2}\right)$	≤ 0
2	$-\frac{v}{(n)_2} + \frac{(v)_2}{2(n)_3}\frac{2\cdot 5+4\cdot 1}{6}$	$=$	$\frac{7v}{6(n)_2}\left(-\frac{6}{7}+\frac{v-1}{n-2}\right)$	
3	$-\frac{3v}{2(n)_2} + \frac{(v)_2}{2(n)_3}\frac{2\cdot 5+4\cdot 3}{6}$	$=$	$\frac{11v}{6(n)_2}\left(-\frac{9}{11}+\frac{v-1}{n-2}\right)$	
4	$-\frac{2v}{(n)_2} + \frac{(v)_2}{2(n)_3}\frac{6\cdot 5}{6}$	$=$	$\frac{5v}{2(n)_2}\left(-\frac{4}{5}+\frac{v-1}{n-2}\right)$	

Table 3.4: Maximum value of $\epsilon(n,w,G,H_1)$.

Table 3.5 gives the bounds on H_0. Note that $\epsilon(n,w,G,H_0) \leq 0$ or v is very large in which case the maximum occurs when $|Y_w \cap H| = 4$.

$\lvert Y_w \cap H\rvert$	maximum value of $\epsilon(n,w,G,H_0)$
1	$-\frac{v}{2(n)_2}+\frac{(v)_2}{2(n)_3}\frac{2\cdot4-2\cdot3}{6} = \frac{v}{2(n)_2}\left(-\frac{1}{2}+\frac{1}{2}\frac{v-1}{n-2}\right) \leq 0$
2	$-\frac{v}{(n)_2}+\frac{(v)_2}{2(n)_3}\frac{2\cdot4+4\cdot1-2\cdot1}{6} = \frac{7v}{6(n)_2}\left(-\frac{6}{7}+\frac{v-1}{n-2}\right)$
3	$-\frac{3v}{2(n)_2}+\frac{(v)_2}{2(n)_3}\frac{2\cdot4+4\cdot3}{6} = \frac{2v}{(n)_2}\left(-\frac{3}{4}+\frac{v-1}{n-2}\right)$
4	$-\frac{2v}{(n)_2}+\frac{(v)_2}{2(n)_3}\frac{6\cdot6}{6} = \frac{3v}{(n)_2}\left(-\frac{2}{3}+\frac{v-1}{n-2}\right)$

Table 3.5: Maximum value of $\epsilon(n,w,G,H_0)$.

Combining all the information contained within the figures we attain our result. This concludes the proof of Claim 42. $\square$

Observe that if we fix n, G and W, we have expressed $f(n,G,W)$ as $\binom{v}{4}^{-1}\sum_{H\in\mathcal{H}_4} d(H)$ for the density function $d(H)$ found in equation (3.4). Then using the bounds for $\epsilon(n,w,G,H)$ in Claim 42 we get upper bound for $d(H)$, call these upper bounds $d^*(H)$. We place in Table 3.6 said values for all $H \in \mathcal{H}_4$.

H	$d^*(H)$			
H_0	$\frac{1}{4}\frac{(v)_4}{(n)_4}$	$+\frac{(n-v)v}{(n)_2}$	$-\frac{(n-v)(v)_2}{(n)_3}$	$+\frac{3(n-v)v}{(n)_2}\max\left\{0,\frac{v-1}{n-2}-\frac{2}{3}\right\}$
H_1	$\frac{1}{6}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v}{(n)_2}$	$-\frac{5(n-v)(v)_2}{6(n)_3}$	$+\frac{5(n-v)v}{2(n)_2}\max\left\{0,\frac{v-1}{n-2}-\frac{4}{5}\right\}$
$H_\wedge$	$\frac{1}{12}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v}{(n)_2}$	$-\frac{2(n-v)(v)_2}{3(n)_3}$	$+\frac{5(n-v)v}{3(n)_2}\max\left\{0,\frac{v-1}{n-2}-\frac{9}{10}\right\}$
$H_\parallel$	$\frac{1}{3}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v}{(n)_2}$	$-\frac{2(n-v)(v)_2}{3(n)_3}$	
$H_\dashv$	$-\frac{3}{4}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v}{(n)_2}$	$-\frac{(n-v)(v)_2}{2(n)_3}$	
$H_\sqcup$		$\frac{(n-v)v}{(n)_2}$	$-\frac{(n-v)(v)_2}{2(n)_3}$	
$H_\triangle$	$\frac{1}{4}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v}{(n)_2}$	$-\frac{(n-v)(v)_2}{2(n)_3}$	
$H_\square$	$-\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v}{(n)_2}$	$-\frac{(n-v)(v)_2}{3(n)_3}$	
H_Q	$-\frac{1}{4}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v}{(n)_2}$	$-\frac{(n-v)(v)_2}{3(n)_3}$	
H_5	$-\frac{1}{2}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v}{(n)_2}$	$-\frac{(n-v)(v)_2}{6(n)_3}$	
H_6		$\frac{(n-v)v}{(n)_2}$		

Table 3.6: The values of $d^*(H)$ for the eleven distinct nonisomorphic graphs on 4 vertices.

We now discuss different cases based upon v and n.

Case 1. $4 \leq v \leq (2n-1)/3$.

In this case all expressions involving the max function are zero and we can simplify all the values in Table 3.6 to the values in Table 3.7.

H	$d^*(H)$
H_0	$\frac{(n-v)v}{(n)_2} \quad +\frac{1}{4}\frac{(v)_4}{(n)_4} - \frac{(n-v)(v)_2}{(n)_3}$
H_1	$\frac{(n-v)v}{(n)_2} \quad +\frac{1}{6}\frac{(v)_3}{(n)_3} - \frac{5(n-v)(v)_2}{6(n)_3}$
$H_\wedge$	$\frac{(n-v)v}{(n)_2} \quad +\frac{1}{12}\frac{(v)_3}{(n)_3} - \frac{2(n-v)(v)_2}{3(n)_3}$
$H_\parallel$	$\frac{(n-v)v}{(n)_2} \quad +\frac{1}{3}\frac{(v)_3}{(n)_3} - \frac{2(n-v)(v)_2}{3(n)_3}$
$H_\dashv$	$\frac{(n-v)v}{(n)_2} \quad -\frac{3}{4}\frac{(v)_3}{(n)_3} - \frac{(n-v)(v)_2}{2(n)_3}$
$H_\sqcup$	$\frac{(n-v)v}{(n)_2} \quad -\frac{(n-v)(v)_2}{2(n)_3}$
$H_\triangle$	$\frac{(n-v)v}{(n)_2} \quad +\frac{1}{4}\frac{(v)_3}{(n)_3} - \frac{(n-v)(v)_2}{2(n)_3}$
$H_\square$	$\frac{(n-v)v}{(n)_2} \quad -\frac{(v)_3}{(n)_3} - \frac{(n-v)(v)_2}{3(n)_3}$
H_Q	$\frac{(n-v)v}{(n)_2} \quad -\frac{1}{4}\frac{(v)_3}{(n)_3} - \frac{(n-v)(v)_2}{3(n)_3}$
H_5	$\frac{(n-v)v}{(n)_2} \quad -\frac{1}{2}\frac{(v)_3}{(n)_3} - \frac{(n-v)(v)_2}{6(n)_3}$
H_6	$\frac{(n-v)v}{(n)_2}$

Table 3.7: The values of $d^*(H)$ in the case of $4 \leq v \leq (2n-1)/3$.

First observe that if $H \in \{H_\dashv, H_\sqcup, H_\square, H_Q, H_5, H_6\}$ we have that $d^*(H) \leq \frac{(n-v)v}{(n)_2}$ simply by dropping the negative terms. We then see that the remaining terms simplify as follows:

$$
\begin{aligned}
d^*(H_0) &\leq \tfrac{(n-v)v}{(n)_2} + \tfrac{(v)_2}{4(n)_4}\left(v^2 + 4nv - 17v - 4n^2 + 12n + 6\right), \\
d^*(H_1) &\leq \tfrac{(n-v)v}{(n)_2} + \tfrac{(v)_2}{6(n)_3}(6v - 5n - 2), \\
d^*(H_\wedge) &\leq \tfrac{(n-v)v}{(n)_2} + \tfrac{(v)_2}{12(n)_3}(9v - 8n - 2), \\
d^*(H_\parallel) &\leq \tfrac{(n-v)v}{(n)_2} + \tfrac{(v)_2}{3(n)_3}(3v - 2n - 2), \\
d^*(H_\triangle) &\leq \tfrac{(n-v)v}{(n)_2} + \tfrac{(v)_2}{4(n)_3}(3v - 2n - 2).
\end{aligned}
$$

Using the assumption that $4 \le v \le (2n-1)/3$ it is simple arithmetic to check that $d^*(H) \le \frac{(n-v)v}{(n)_2}$ for $H \in \{H_1, H_\wedge, H_\|, H_\triangle\}$. The remaining case is $H = H_0$. In regards to H_0 observe that the expression $v^2 + 4nv - 17v - 4n^2 + 12n + 6$ increases in v for any fixed $n \ge 4$. Hence, this expressions maximum occurs when $v = (2n-1)/3$ and is at most $(-8n^2 - 10n + 106)/9$, which is negative when $n \ge 4$. Therefore

$$f(n, G, W) \le \max\{d^*(H) : H \in \mathcal{H}_4\} \le \frac{(n-v)v}{(n)_2} \le \frac{1}{(n)_2}\left\lfloor \frac{n^2}{4} \right\rfloor.$$

Case 2. $v \ge 2n/3$ (and $v \ge 4$).

Observe that since $v \ge 2n/3$, some of the expressions in Table 3.7 now have values greater than $\frac{(n-v)v}{(n)_2}$. We will then use flag algebras to provide the means to "balance" out some of the negative terms, in say H_6, with the positive terms, in say $H_\|$. Recall that $N(v)$ denotes the neighborhood of the vertex v and $N[v] = N(v) \cup \{v\}$ to denote the closed neighborhood of the vertex v. Note that since we only consider simple graphs $N(v) \ne N[v]$. For any vertex subset S, we denote its complement $\overline{S} := V - S$. We proceed by adding a nonnegative term to our function. This is the term that arises from the flag algebra calculations. The coefficient $\gamma \ge 0$ will be chosen later. Now

$$\begin{aligned}
f(n, G, W) &\le \frac{1}{\binom{v}{4}} \sum_{H \in \mathcal{H}_4} d^*(H) \\
&\le \frac{1}{\binom{v}{4}} \sum_{H \in \mathcal{H}_4} d^*(H) \\
&\quad + \frac{\gamma}{\binom{v}{4}} \sum_{(z_1,z_2): z_1z_2 \in E(\overline{G})} \Big(|N(z_1) \cap \overline{N[z_2]}| - |\overline{N[z_1]} \cap N(z_2)| \Big)^2 && (3.5) \\
&\quad + \frac{\gamma}{\binom{v}{4}} \sum_{(z_1,z_2): z_1z_2 \in E(G)} \Big(|N(z_1) \cap N(z_2)| - |\overline{N[z_1]} \cap \overline{N[z_2]}| \Big)^2 && (3.6)
\end{aligned}$$

The motivation behind where these equations come from will be given in subsection 3.3.1.

The values associated with H from (3.5) gives a value of -16γ for each subgraph isomorphic to $H_\|$, since it assigns a value of -2γ for each ordered pair of distinct nonadjacent vertices. Also the expression (3.6) assigns $+8\gamma$ for each subgraph isomorphic to $H_\|$, because it assigns a value of $+2\gamma$ for each ordered pair of distinct adjacent vertices. Together the net value for $H_\|$ is -8γ. All of the computed values for each of the subgraphs is contained in Table 3.8.

H	$\gamma c(H)$				
H_0	0	$+$	0	$=$	0
H_1	0	$+$	4γ	$=$	4γ
$H_\wedge$	4γ	$+$	0	$=$	4γ
$H_{\|\|}$	-16γ	$+$	8γ	$=$	-8γ
$H_\dashv$	0	$+$	0	$=$	0
$H_\sqcup$	-4γ	$+$	0	$=$	-4γ
$H_\triangle$	12γ	$+$	-12γ	$=$	0
$H_\square$	0	$+$	0	$=$	0
H_Q	0	$+$	-4γ	$=$	-4γ
H_5	0	$+$	4γ	$=$	4γ
H_6	0	$+$	24γ	$=$	24γ

Table 3.8: The values of $\gamma c(H)$ for the eleven distinct nonisomorphic graphs on 4 vertices. The first term is the contribution from the expression (3.5), the second term from the expression in (3.6). Their sum is the last term.

We then have the expression

$$f(n,G,W) \leq \frac{1}{\binom{v}{4}} \sum_{H\in\mathcal{H}_4} d^*(H) + \frac{\gamma}{\binom{v}{4}} \sum_{H\in\mathcal{H}_4} c(H) + \frac{\gamma}{\binom{v}{4}}(6\alpha_1(G) + 6\alpha_3(G)), \tag{3.7}$$

$$\leq \frac{1}{\binom{v}{4}} \sum_{H\in\mathcal{H}_4} (d^*(H) + \gamma c(H)) + \frac{24\gamma}{v-3}. \tag{3.8}$$

We now choose the value of γ so we can insure that if we limit n to infinity we have that $d^*(H) + \gamma c(H) = \frac{(n-v)v}{(n)_2} + 24\gamma$ is exactly 1/4. Hence we set $\gamma = \frac{1}{96} - \frac{(n-v)v}{24(n)_2}$. Since $v \geq 2n/3$ we have that $\gamma \geq 0$ for all $n \geq 7$. It remains to show that $d^*(H) + \gamma c(H) \leq 1/4$ for all choices of H. We list all the expressions for $d^*(H) + \gamma c(H)$ in Table 3.9.

H	$d^*(H) + \gamma c(H) - 1/4$		
H_0	$-\frac{1}{4}$	$+\frac{1}{4}\frac{(v)_4}{(n)_4}$	$+\frac{(n-v)v(2v-n)}{(n)_3}$
H_1	$-\frac{5}{24}$	$+\frac{1}{6}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v}{6(n)_3}\max\{5n-5v-5, 10v-7n+4\}$
$H_\wedge$	$-\frac{5}{24}$	$+\frac{1}{12}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v}{6(n)_3}\max\{5n-4v-6, 6v-4n+2\}$
$H_\parallel$	$-\frac{1}{3}$	$+\frac{1}{3}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v(4n-2v-6)}{3(n)_3}$
$H_\dashv$	$-\frac{1}{4}$	$-\frac{3}{4}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v(2n-v-3)}{2(n)_3}$
$H_\sqcup$	$-\frac{7}{24}$		$+\frac{(n-v)v(7n-3v-11)}{6(n)_3}$
$H_\triangle$	$-\frac{1}{4}$	$+\frac{1}{4}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v(2n-v-3)}{2(n)_2}$
$H_\square$	$-\frac{1}{4}$	$-\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v(3n-v-5)}{3(n)_3}$
H_Q	$-\frac{7}{24}$	$-\frac{1}{4}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v(7n-2v-12)}{6(n)_3}$
H_5	$-\frac{5}{24}$	$-\frac{1}{2}\frac{(v)_3}{(n)_3}$	$+\frac{(n-v)v(5n-v-9)}{6(n)_3}$
H_6	0		

Table 3.9: Simplified values of $d^*(H) + \gamma c(H) - 1/4$ in the case of $v \geq 2n/3$.

We then simplify the expressions in Table 3.9 and subtract 1/4 from them to obtain the equations in Table 3.10. Our goal will be to show that all the expressions in Table 3.10 are ≤ 0 for each of the eleven nonisomorphic graphs H on 4 vertices. We express the functions in Tables 3.10 in terms of the variable $x = v/n$ to simplify the the expressions and ignore the lower order terms as these terms will be included in the $o(1)$ term. Label each of these functions as $g_H(x)$. The last column in Table 3.10 consists of using calculus to find the maximum value in over the domain.

H	$g_H(x)$	max over $\frac{2}{3} \le x \le 1$
H_0	$-\frac{1}{4} + \frac{1}{4}x^4 + (1-x)x(2x-1)$	0 @ $x = 1$
H_1	$-\frac{5}{24} + \frac{x^3}{6} + \frac{(1-x)x}{6} \max\{5-5x, 10x-7\}$	$-\frac{1}{24}$ @ $x = 1$
$H_\wedge$	$-\frac{5}{24} + \frac{x^3}{12} + \frac{(1-x)x}{6} \max\{5-4x, 6x-4\}$	$-\frac{7}{72}$ @ $x = \frac{2}{3}$
$H_\parallel$	$-\frac{1}{3} + \frac{x^3}{3} + \frac{(1-x)x(4-2x)}{3}$	0 @ $x = 1$
$H_\dashv$	$-\frac{1}{4} - \frac{3x^3}{4} + \frac{(1-x)x(2-x)}{2}$	$-\frac{35}{108}$ @ $x = \frac{2}{3}$
$H_\sqcup$	$-\frac{7}{24} + \frac{(1-x)x(7-3x)}{6}$	$-\frac{23}{216}$ @ $x = \frac{2}{3}$
$H_\triangle$	$-\frac{1}{4} + \frac{x^3}{4} + \frac{(1-x)x(2-x)}{2}$	0 @ $x = 1$
$H_\square$	$-\frac{1}{4} - x^3 + \frac{(1-x)x(3-x)}{3}$	$-\frac{121}{324}$ @ $x = \frac{2}{3}$
H_Q	$-\frac{7}{24} - \frac{x^3}{4} + \frac{(1-x)x(7-2x)}{6}$	$-\frac{101}{648}$ @ $x = \frac{2}{3}$
H_5	$-\frac{7}{24} - \frac{x^3}{2} + \frac{(1-x)x(5-x)}{6}$	$-\frac{127}{648}$ @ $x = \frac{2}{3}$
H_6	0	0 @ $x \in [2/3, 1]$

Table 3.10: The $g_H(x)$ functions that determine the asymptotic value of $d^*(H) + \gamma c(H) - 1/4$.

The equations in Table 3.10 are all nonpositive on the domain and hence we have that

$$\lim_{n\to\infty} f(n, G, W) \le \frac{1}{4}.$$

This concludes the proof of Lemma 40.

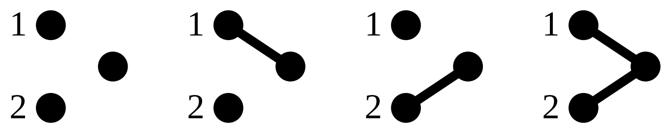
□

3.3.1 Flag Algebra Motivation

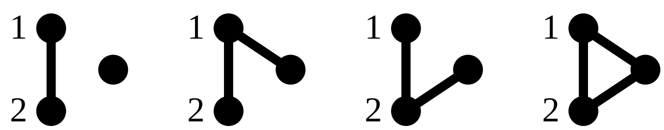
Observe that the summations (3.5) and (3.6) sum over the nonedges and edges respectively. This is similar to the idea in flag algebras we defined in Chapter 2, in which some of our calculations is to sum over different types. In fact that is the very idea here.

Define $\sigma_1 = \overline{K_2}$ and $\sigma_2 = K_2$. These will be the two different types in our flag algebra calculations. Let $\mathcal{H}_4$ be the set of all four vertex subgraphs, with the notation described in section 3.3. Let $\mathcal{F}_3^{\sigma_1}$ and $\mathcal{F}_3^{\sigma_2}$ be all the three vertex subgraphs that correspond to the types

σ_1 and σ_2 respectively. We then label the flags in $\mathcal{F}_3^{\sigma_1}$ and $\mathcal{F}_3^{\sigma_2}$ as follows: If $F \in \mathcal{F}_3^{\sigma_1}$ contains no edge denote F by $\overline{F_1}$; If $F \in \mathcal{F}_3^{\sigma_1}$ contains exactly one edge incident to the vertex labeled 1 under σ_1 then denote F by $\overline{F_2}$; If $F \in \mathcal{F}_3^{\sigma_1}$ contains exactly one edge incident to the vertex labeled 2 under σ_1 then denote F by $\overline{F_3}$; If $F \in \mathcal{F}_3^{\sigma_1}$ contains exactly two edges then denote F by $\overline{F_4}$. Similarly denote the flags in $\mathcal{F}_3^{\sigma_2}$ as F_1, F_2, F_3 and F_4. We offer the following figure as a visual aid to these definitions.



(a) $\overline{F_1}$ (b) $\overline{F_2}$ (c) $\overline{F_3}$ (d) $\overline{F_4}$



(e) F_1 (f) F_2 (g) F_3 (h) F_4

Figure 3.1: The flags contained in $\mathcal{F}_3^{\sigma_1}$ and $\mathcal{F}_3^{\sigma_2}$

At this point, the standard approach to flag algebras would be to calculate all the probabilities of these flags occurring within each subgraph in $\mathcal{H}_4$, however the number of these are based on v and n. This poses a problem, as a standard SDP solver cannot handle unknown parameters within the computations. Therefore we must follow a similar method to what we used in the proof of Mantel's Theorem in section 2.2. Note that we have four flags associated with both σ_1 and σ_2 so our semidefinite matrices will be 4×4 matrices. Let $\overline{Q} = [\overline{q_{ij}}]$ and $Q = [q_{ij}]$ be the semidefinite matrices corresponding to σ_1 and σ_2 respectively. It is important to note that $\overline{Q}$ and Q are only over the matrices and signify complements in anyway. First the case $n = v$ as that is when the density of $H_{\|}$ is highest, namely 1/3. After formulating all the equations involving c_H for all $H \in \mathcal{H}_4$ we observe that $c_{H_{\|}} = \frac{1}{3}q_{11} + \frac{1}{3}\overline{q_{32}} + \frac{1}{3}\overline{q_{23}}$ and $c_{H_6} = q_{44}$. Since the density of $H_{\|}$ is 1/3 and the density of H_6 is 0 we set $c_{H_{\|}} = -1/12$ and $c_{H_6} = 1/4$.

Then solving for all of the variables we get the matrices

$$\overline{Q} = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad Q = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

At this point transitioning back to the case where we allow $v \neq n$ would normally be an issue in most cases, but here we make an educated guess and allow the constant $1/4$ to be perturbed while keeping the same structure. (So in a sense, we guessed.) This gave us the matrices

$$\overline{Q} = \gamma\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad Q = \gamma\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

where $\gamma = \frac{1}{8} - \frac{(n-v)v}{2(n)_2}$.

This value of γ would complete our proof. In an effort to make the proof more readable we transitioned out of flag algebras. This is why the sums given in the proof of Lemma 40 seem to come out of nowhere. First observe that the total array of our calculation is

$$\sum_{H \in \mathcal{H}_4} c_H = \sum_{H \in \mathcal{H}_4} \left(\gamma[p(\overline{F_i}, \theta; H)]^T \overline{Q}[p(\overline{F_i}, \theta; H)] + \gamma[p(F_i, \theta; H)]^T Q[p(F_i, \theta; H)]\right).$$

In order to transition into the notation we will look specifically at Q. Now Q corresponds to the type with an edge. This will correspond the to the sum over the edges. Observe that since $p(F_i, \theta; H)$ is chosen uniformly we have that

$$p(F_i, F_j, \theta; H) = \frac{\text{Number of } F_i, F_j \text{ in } H \text{ that only intersect on } \theta}{24}.$$

We may then factor $1/24$ out and place it in γ. This makes $\gamma = \frac{1}{96} - \frac{(n-v)v}{24(n)_2}$, our γ in the proof of Lemma 40. Our only concern is describing the number of F_i times the number of F_j according to our matrix Q. But this is precisely the number of times the neighborhoods of the vertices, z_1, z_2 incident to the edge in σ_1 intersect according to $\left(|N(z_1) \cap \overline{N[z_2]}| - |\overline{N[z_1]} \cap N(z_2)|\right)^2$. We have a similar result using $\overline{Q}$. Then summing over $\mathcal{H}_4$ but keeping the equations corresponding to

σ_1 and σ_2 seperate give rise the the expressions (3.5) and (3.6). This completes the motivation behind expressions (3.5) and (3.6).

CHAPTER 4. Other Results

4.1 The 3 Crown

Recall from Chapter 1 that Griggs and Lu [GL09] and Lu [Lu14] solved the asymptotic case for the k-crown for all values of $k \neq 3, 5$. In Griggs and Lu [GL09] they did provide an general upper bound of

$$|\mathcal{F}| \leq \left(1 + \frac{\sqrt{2}}{2} + o(1)\right)\binom{n}{\lfloor n/2 \rfloor}$$

when $\mathcal{F}$ lives in the Boolean lattice. We consider the simplified problem where $\mathcal{F}$ is restricted to existing on two consecutive layers.

Let the six crown $\mathcal{C}_6$ be the poset with sets A, B, C, D, E, F such that $A \subset D, E$ and $B \subset E, F$ and $C \subset D, F$.

Theorem 43 *Let $\mathcal{F}$ be a C_6-free family of the Boolean lattice existing on only two layers. Then $|\mathcal{F}| \leq (2\sqrt{3} - 2)\binom{n}{\lfloor n/2 \rfloor}$.*

Proof. Without loss of generality let $\mathcal{F}$ be on the layers consisting of sets of size $\binom{n}{k}$ and $\binom{n}{k+1}$. Let $\mathcal{S}$ be the sets in $\mathcal{F}$ of size $\binom{n}{k}$ and $\mathcal{U}$ be the sets in $\mathcal{F}$ of size $\binom{n}{k+1}$. Let $\mathcal{X}$ be the layer of the Boolean lattice one layer below $\binom{n}{k}$, i.e. $\mathcal{X}$ consists of all sets of size $\binom{n}{k-1}$. For all $X \in \mathcal{X}$ let $T(X)$ denote the sets $U \in \mathcal{F}$ such that $X \subset U$ and $|U| = \binom{n}{k+1}$. Similarly let $B(X)$ denote the sets $S \in \mathcal{F}$ such that $X \subset S$ and $|S| = \binom{n}{k}$.

Define the X-graph as follows. Let $V(X) = \{v \in \mathcal{S} : X \in \mathcal{X}, X \subset v\}$ and $E(X) = \{e \in \mathcal{U} : X \in \mathcal{X}, X \subset e\}$.

Claim 44 *There is a 3-cycle in some X-graph if and only if there is a $\mathcal{C}_6$ in the family.*

Proof of Claim 44 First we prove that given a triangle in an X-graph there is a corresponding $\mathcal{C}_6$ in the family of posets. Let $X \in \mathcal{X}$ such that the X-graph contains a triangle T. Let this triangle be denoted by $v_1, e_1, v_2, e_2, v_3, e_3, v_1$ where $v_1, v_2, v_3 \in V(x)$ and $e_1, e_2, e_3 \in E(x)$. By definition $v_1, e_1, v_2, e_2, v_3, e_3, v_1 \in \mathcal{F}$ and then $v_1, e_1, v_2, e_2, v_3, e_3, v_1$ gives a $\mathcal{C}_6$ in $\mathcal{F}$.

Next we show that a $\mathcal{C}_6$ in $\mathcal{F}$ produces a triangle in some X-graph. Let v_1, v_2, v_3 and e_1, e_2, e_3 be the sets in $\mathcal{C}_6$ of size $\binom{n}{k}$ and $\binom{n}{k+1}$ respectively such that $v_1 \subset e_1, e_2$, $v_2 \subset e_2, e_3$ and $v_3 \subset e_1, e_3$.

Now we show that v_1, v_2, v_3 exists in some X-graph. Suppose no $X \in \mathcal{X}$ exists such that $X \subset v_1, v_2, v_3$. Observe that $v_1 = e_1 \cap e_2$ since our sets exist on adjacent layers. Hence the size of the symmetric difference of e_1 and e_2 is 2. Similarly $|e_2 \triangle e_3| = |e_1 \triangle e_3| = 2$. Also since $v_1, v_2 \subset e_1$ and their layers are adjacent, $|v_1 \triangle v_2| = 2$. Similarly $|v_2 \triangle v_3| = |v_1 \triangle v_3| = 2$.

Since $|v_1| = |v_2| = k$ and $|v_1 \triangle v_2| = 2$ there exists $X' \in \mathcal{X}$ such that $v_1 = X' \cup \{a\}$, $v_2 = X' \cup \{b\}$ where $a \neq b$, $a \notin v_2$, and $b \notin v_1$. If $v_3 = X' + \{c\}$ for some $c \in [n]$ then v_1, v_2, v_3 are vertices in the X'-graph. (Note that c must not be in X' since $|v_3| = \binom{n}{k}$ and $c \neq a, b$ else $v_1 = v_3$ or $v_2 = v_3$ which contradicts the symmetric difference requirement.)

Suppose no $c \in [n]$ exists such that $v_3 = X' + \{c\}$. Since $|v_3| - |X'| = 1$ there must be an element $d \in X'$ such that $d \notin v_3$. Then $d \in v_1, v_2$. Note that $d \neq a, b$. Since $|v_1 \triangle v_2| = 2$ and $|v_2 \triangle v_3| = 2$ and $|v_1| = |v_2| = |v_3|$ we must have that $v_1 - v_3 = \{d\}$, $v_2 - v_3 = \{d\}$ and there are $f, g \in [n], f \neq g$ such that $v_3 - v_1 = \{f\}$ and $v_3 - v_2 = \{g\}$. Note however that $g \in v_1$ and $f \in v_2$ else $|v_3 - v_1| \geq 2$ and $|v_3 - v_2| \geq 2$ respectively. Hence $|v_1 - v_2| + |v_2 - v_1| \geq |\{a, g\}| + |\{b, f\}| > 2$ if either $f \neq b$ or $g \neq a$. Consider e_2 with $v_2, v_3 \subset e_2$ and $|e_2| = k+1$. Hence $X' \cup \{b\} = v_2 \subset e_2$ and $a \in e_2$ since $v_3 \subset e_2$. Since $a \notin v_2$ we have that $e_2 = X' \cup \{a, b\} = e_1$ a contradiction. Therefore v_1, v_2, v_3 are in the X'-graph. Hence e_1, e_2, e_3 are edges in the X'-graph and $v_1 e_1 v_2 e_2 v_3 e_3 v_1$ forms a 3-cycle in the X'-graph.

This completes the proof of Claim 44. $\square$

We now return to the proof of our theorem. We are going to put an upper bound on the size of $T(X)$. A result of Turán's [Tur41] and by Claim 44 we can have at most $\frac{|B(X)|^2}{4}$ sets in $T(X)$ that contain two elements of $B(X)$.

Next we observe that each pair of sets $A, B \in \binom{n}{k}$ such that $X = A \cap B$ has a one to one correspondence to each set $C \in \binom{n}{k+1}$ such that $X \subset C$. We can see this from the following.

Define sets $\mathcal{A} = \{A \in \binom{n}{k} : X \subset A\}$ and $\mathcal{C} = \{C \in \binom{n}{k+1} : X \subset C\}$. Then define a function $f_X : \mathcal{A} \times \mathcal{A} \to \mathcal{C}$ such that if $A \neq B$ and $A \cap B = X$ then $f_X(A, B) = A \cup B$ and undefined otherwise. This function is onto since $C - X = \{a, b\}$, $a \neq b$ and we may set $A = X \cup \{a\}$ and $B = X \cup \{b\}$ which causes $f_X(A, B) = C$. This function is one to one. Suppose $f_X(A, B) = C$ and $f_X(A', B') = C$. Now $C - X = \{a, b\}$. Since $X \subset A, B$ by definition we have that without loss of generality $A = X \cup \{a\}$ and $B = X \cup \{b\}$. But then a similar argument gives $A' = X \cup \{a\}, B' = X \cup \{b\}$ or $A' = X \cup \{b\}, B' = X \cup \{a\}$ which means that $(A, B) = (A', B')$ as pairs of sets.

Then by naively over counting the rest of the sets in $T(x)$ by counting each pair of sets that do not consist of two sets in $B(x)$ we have that

$$\begin{aligned} |T(x)| &\leq \frac{|B(x)|^2}{4} + |B(x)|\,(n-k+1-|B(x)|) + \frac{(n-k+1-|B(x)|)\,(n-k-|B(x)|)}{2} \\ &\leq \frac{|B(x)|^2}{4} + |B(x)|\,(n-k+1-|B(x)|) + \frac{(n-k+1-|B(x)|)^2}{2} \\ &= \frac{1}{2}(n-k+1)^2 - \frac{|B(x)|^2}{4}. \end{aligned} \tag{4.1}$$

Let $\mathcal{Y}$ be the layer of the boolean lattice one layer above $\binom{n}{k+1}$, i.e. $\mathcal{Y}$ consists of all sets of size $\binom{n}{k+2}$. For all $Y \in \mathcal{Y}$ let $T'(Y)$ denote the sets $U \in \mathcal{F}$ such that $U \subset Y$ and $|U| = \binom{n}{k+1}$. Similarly let $B'(Y)$ denote the sets $S \in \mathcal{F}$ such that $S \subset U$ and $|S| = \binom{n}{k}$. Due to symmetry we have that $|B'(y)| \leq \frac{1}{2}(k+1)^2 - \frac{|T'(Y)|}{4}$.

Observe that $k^{-1}\sum_{X\in\mathcal{X}} |B(X)| = \binom{n-k}{2}^{-1} \sum_{Y\in\mathcal{Y}} |B'(Y)| = |\mathcal{S}|$ and $\binom{k+1}{2}^{-1} \sum_{X\in\mathcal{X}} |T(X)| = (n-k-1)^{-1}\sum_{Y\in\mathcal{Y}} |T'(Y)| = |\mathcal{U}|$. Then by a counting argument, (4.1), and the Cauchy-Schwarz inequality we have

$$\begin{aligned} \binom{k+1}{2}|\mathcal{U}| &= \sum_{X\in\mathcal{X}} |T(X)| \\ &\leq \sum_{X\in\mathcal{X}} \left[\frac{(n-k+1)^2}{2} - \frac{|B(X)|^2}{4}\right] \\ &= \frac{(n-k+1)^2}{2}\binom{n}{k-1} - \frac{1}{4}\sum_{X\in\mathcal{X}} |B(X)|^2 \\ &\leq \frac{(n-k+1)^2}{2}\binom{n}{k-1} - \frac{1}{4}\frac{k^2|\mathcal{S}|^2}{\binom{n}{k-1}}. \end{aligned} \tag{4.2}$$

And similarly we have that

$$\binom{n-k}{2}|\mathcal{S}| \leq \frac{(k+1)^2}{2}\binom{n}{k+2} - \frac{1}{4}\frac{(n-k-1)^2|\mathcal{U}|^2}{\binom{n}{k+2}}. \tag{4.3}$$

Set $b = |\mathcal{S}|/\binom{n}{k}$ and $t = |\mathcal{U}|/\binom{n}{k}$. We wish to maximize $|\mathcal{U}| + |\mathcal{S}|$ but this is equivalent to maximizing $t + b$. Observe that performing some algebra on (4.2) and (4.3) gives us

$$t \leq \frac{(n-k+1)}{(k+1)} - \frac{1}{2}\frac{(n-k+1)}{(k+1)}b^2 \tag{4.4}$$
$$b \leq \frac{(k+1)}{(k+2)} - \frac{1}{2}\frac{(k+2)(k+1)}{(n-k)^2}t^2 \tag{4.5}$$

This problem is then the following maximization problem.

$$\text{Maximize} \quad t + b$$

$$\begin{aligned} \text{Subject to :} \quad t &\leq \frac{(n-k+1)}{(k+1)} - \frac{1}{2}\frac{(n-k+1)}{(k+1)}b^2 \\ b &\leq \frac{(k+1)}{(k+2)} - \frac{1}{2}\frac{(k+2)(k+1)}{(n-k)^2}t^2 \end{aligned}$$

Observe that since the objective function is a nonnegative linear equation of two variables and the constraints are upper bounds we may assume that one of the two constraints is realized otherwise we may just increase one of the variables until a constraint is realized.

We now show that the solution to the maximization problem is the intersection of the equations $t = \frac{(n-k+1)}{(k+1)} - \frac{1}{2}\frac{(n-k+1)}{(k+1)}b^2$ and $b = \frac{(k+1)}{(k+2)} - \frac{1}{2}\frac{(k+2)(k+1)}{(n-k)^2}t^2$. Without loss of generality suppose we are on the constraint $t = \frac{(n-k+1)}{(k+1)} - \frac{1}{2}\frac{(n-k+1)}{(k+1)}b^2$. Suppose $b_1 - b_2 = \epsilon$. By the derivative of t, $t' = -\frac{(n-k+1)}{(k+1)}b$, we have that decreasing b_1 to b_2 t will be increased by more than $t'_1 > t'_2 = -\frac{(n-k+1)}{(k+1)}b_2 = \frac{(n-k+1)}{(k+1)}b_1$ due to the concavity of the constraint. Hence we want b to be as small as possible along this constraint. We may do this as long as the second constraint is still a strict inequality. Similarly if we are on the constraint $b = \frac{(k+1)}{(k+2)} - \frac{1}{2}\frac{(k+2)(k+1)}{(n-k)^2}t^2$ we want t to be as small as possible which we may do as long as the first constraint is a strict inequality. Therefore the solution to the optimization problem is where the constraints intersect.

Now by Claim 33 we know that $k \in [n/2 - n^{2/3}, n/2 + n^{2/3}]$. Hence for any constants p, q

we have that $\frac{k+p}{n-k+q} \to 1$ as $n \to \infty$. Therefore asymptotically these equations are

$$t = 1 - \frac{1}{2}b^2$$
$$b = 1 - \frac{1}{2}t^2$$

Then $t = b = \sqrt{3} - 1$ and the result follows. $\square$

BIBLIOGRAPHY

[ACC+94] Paul D. Amer, Christophe Chassot, Thomas J. Connolly, Michel Diaz, and Phillip Conrad. Partial-order transport service for multimedia and other applications. *IEEE/ACM Transaction on Networking*, 1994.

[AMM11] M. Axenovich, J. Manske, and M Martin. $\mathcal{Q}_2$-free families of the boolean lattice. *Order published online*, 2011.

[BFK+12] Bartłomiej Bosek, Stefan Felsner, Kamil Kloch, Tomasz Krawczyk, Grzegorz Matecki, and Piotr Micek. On-line chain partitions of orders: a survey. *Order*, 29(1):49–73, 2012.

[Bol65] B. Bollobás. On generalized graphs. *Acta Math. Acad. Sci. Hungar*, 16:447–452, 1965.

[Bol98] Béla Bollobás. *Modern Graph Theory.* Springer Science+Business Media, Inc., 1998.

[BT11] Rahil Baber and John Talbot. Hypergraphs do jump. *Combin. Probab. Comput.*, 20(2):161–171, 2011.

[Buk09] Boris Bukh. Set families with a forbidden subposet. *Electron. J. Combin.*, 16(1):Research Paper 142, 11, 2009.

[CKM+12] Daniel W. Cranston, William B. Kinnersley, Kevin G. Milans, Gregory J. Puleo, and Douglas B. West. Chain-making games in grid-like posets. *J. Comb.*, 3(4):633–649, 2012.

[CNNS94] A.N. Choudhary, B. Narahari, D.M. Nicol, and R. Simha. Optimal processor assignment for a class of piplined computations. *IEEE Transactions on Parallel and Distributed Systems*, 1994.

[DBK07] Annalisa De Bonis and Gyula O. H. Katona. Largest families without an r-fork. *Order*, 24(3):181–191, 2007.

[DBKS05] Annalisa De Bonis, Gyula O. H. Katona, and Konrad J. Swanepoel. Largest family without $A \cup B \subseteq C \cap D$. *J. Combin. Theory Ser. A*, 111(2):331–336, 2005.

[Die05] Reinhard Diestel. *Graph Theory*. Springer-Verlag Berlin Heidelberg, 2005.

[Erd45] P. Erdős. On a lemma of Littlewood and Offord. *Bull. Amer. Math. Soc.*, 51:898–902, 1945.

[Fel04] Stefan Felsner. Lattice structures from planar graphs. *Electron. J. Combin.*, 11(1):Research Paper 15, 24 pp. (electronic), 2004.

[FK08] S. Felsner and K. B. Knauer. Distributive lattices from graphs. In *Sixth Conference on Discrete Mathematics and Computer Science (Spanish)*, pages 11–23. Univ. Lleida, Lleida, 2008.

[FKPRW00] Stefan Felsner, Ravi Kant, C. Pandu Rangan, and Dorothea Wagner. On the complexity of partial order properties. *Order*, 17(2):179–193, 2000.

[FKT13] Stefan Felsner, Tomasz Krawczyk, and William T. Trotter. On-line dimension for posets excluding two long incomparable chains. *Order*, 30(1):1–12, 2013.

[FLT10] Stefan Felsner, Ching Man Li, and William T. Trotter. Adjacency posets of planar graphs. *Discrete Math.*, 310(5):1097–1104, 2010.

[FRV12] Victor Falgas-Ravry and Emil R. Vaughan. Turán H-densities for 3-graphs. *Electron. J. Combin.*, 19(3):Paper 40, 26, 2012.

[FT05] Stefan Felsner and William T. Trotter. Posets and planar graphs. *J. Graph Theory*, 49(4):273–284, 2005.

[GK08] Jerrold R. Griggs and Gyula O. H. Katona. No four subsets forming an N. *J. Combin. Theory Ser. A*, 115(4):677–685, 2008.

[GL09] Jerrold R. Griggs and Linyuan Lu. On families of subsets with a forbidden subposet. *Combin. Probab. Comput.*, 18(5):731–748, 2009.

[GLL12] J.R. Griggs, W.-T. Li, and L. Lu. Diamond-free families. *J. Combinatorial Theory*, 119, 2012.

[Hog07] Leslie Hogben, editor. *Handbook of Linear Algebra.* Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2007. Associate editors: Richard Brualdi, Anne Greenbaum and Roy Mathias.

[Hun33] Edward V. Huntington. A second correction to: "New sets of independent postulates for the algebra of logic, with special reference to Whitehead and Russell's *principia mathematica*" [Trans. Amer. Math. Soc. **35** (1933), no. 1, 274–304; 1501684]. *Trans. Amer. Math. Soc.*, 35(4):971, 1933.

[JM11] Gwenaël Joret and Kevin G. Milans. First-fit is linear on posets excluding two long incomparable chains. *Order*, 28(3):455–464, 2011.

[Kee11] Peter Keevash. Hypergraph Turán problems. In *Surveys in combinatorics 2011*, volume 392 of *London Math. Soc. Lecture Note Ser.*, pages 83–139. Cambridge Univ. Press, Cambridge, 2011.

[KMY13] Lucas Kramer, Ryan R. Martin, and Michael Young. On diamond-free subposets of the Boolean lattice. *J. Combin. Theory Ser. A*, 120(3):545–560, 2013.

[KT83] G. O. H. Katona and T. G. Tarján. Extremal problems with excluded subgraphs in the n-cube. In *Graph theory (Lagów, 1981)*, volume 1018 of *Lecture Notes in Math.*, pages 84–93. Springer, Berlin, 1983.

[Lu14] Linyuan Lu. On crown-free families of subsets. *Arxiv-Preprint*, 2014+.

[Lub66] D Lubell. A short proof of sperner's lemma. *J of Comb Theory*, 1:299–299, 1966.

[McC97] William McCune. Solution of the Robbins problem. *J. Automat. Reason.*, 19(3):263–276, 1997.

[Meš63] L. D. Mešalkin. A generalization of Sperner's theorem on the number of subsets of a finite set. *Teor. Verojatnost. i Primenen*, 8:219–220, 1963.

[MM00] Heikki Mannila and Christopher Meek. Global partial orders from sequential data. *Proc. 6th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining (KDD 2000)*, 2000.

[Raz] Alexander Razborov. Flag algebras: an interim report. http://people.cs.uchicago.edu/~razborov/files/flag_survey.pdf.

[Raz07] Alexander A. Razborov. Flag algebras. *J. Symbolic Logic*, 72(4):1239–1282, 2007.

[RLG13] Steve Butler Ronald L. Graham, Jaroslav Nešetřil, editor. *Mathematics of Paul Erdős II*. Springer New York Heidelberg Dordrecht London, 2013.

[Spe28] Emanuel Sperner. Ein Satz über Untermengen einer endlichen Menge. *Math. Z.*, 27(1):544–548, 1928.

[Tha98] Hai Tran Thanh. An extremal problem with excluded subposet in the Boolean lattice. *Order*, 15(1):51–57, 1998.

[Tur41] Paul Turán. On an extremal problem in graph theory. *Matematikai s Fizikai Lapok*, 48:436–452, 1941.

[VTL82] Jacobo Valdes, Robert E. Tarjan, and Eugene L. Lawler. The recognition of series parallel digraphs. *SIAM J. Comput.*, 11(2):298–313, 1982.

[Yam54] Koichi Yamamoto. Logarithmic order of free distributive lattice. *J. Math. Soc. Japan*, 6:343–353, 1954.