Tiling tripartite graphs with 3-colorable graphs

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Abstract

For any positive real number \( \gamma \) and any positive integer \( h \), there is \( N_0 \) such that the following holds. Let \( N \geq N_0 \) such that \( N \) is divisible by \( h \). If \( G \) is a tripartite graph with \( N \) vertices in each vertex class such that every vertex is adjacent to at least \((2/3 + \gamma)N\) vertices in each of the other classes, then \( G \) can be tiled perfectly by copies of \( K_{h,h,h} \). This extends the work in [19] and also gives a sufficient condition for tiling by any fixed 3-colorable graph. Furthermore, we show that the minimum-degree \((2/3 + \gamma)N\) in our result cannot be replaced by \(2N/3 + h - 2\).

1 Introduction

Let \( H \) be a graph on \( h \) vertices, and let \( G \) be a graph on \( n \) vertices. Tiling (or packing) problems in extremal graph theory are investigations of conditions under which \( G \) must contain many vertex disjoint copies of \( H \) (as subgraphs), where minimum degree conditions are studied the most. An \( H \)-tiling of \( G \) is a subgraph of \( G \) which consists of vertex-disjoint copies of \( H \). A \textit{perfect} \( H \)-tiling, or \textit{\( H \)-factor}, of \( G \) is an \( H \)-tiling consisting of \( \lfloor n/h \rfloor \) copies of \( H \). A very early tiling result is Dirac’s theorem on Hamilton cycles [6], which implies that every \( n \)-vertex graph \( G \) with minimum degree \( \delta(G) \geq n/2 \) contains a perfect matching (usually called 1-factor, instead of \( K_2 \)-factor). Later Corrádi and Hajnal [4] studied the minimum degree of \( G \) that guarantees a \( K_3 \)-factor. Hajnal and Szemerédi [9] settled the tiling problem for any complete graph \( K_h \) by showing that every \( n \)-vertex graph \( G \) with \( \delta(G) \geq (h-1)n/h \) contains a \( K_r \)-factor (it is easy to see that this is sharp). Using the celebrated Regularity Lemma of Szemerédi [22], Alon and Yuster [1, 2] generalized the above tiling results for arbitrary \( H \). Their theorems were later sharpened by various researchers [14, 12, 21, 17]. Results and methods for tiling problems can be found in a recent survey of Kühn and Osthus [18].

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In this paper, we consider multipartite tiling, which restricts $G$ to be an $r$-partite graph. When $r = 2$, The König-Hall Theorem (e.g. see [3]) answers the 1-factor problem for bipartite graphs. Wang [23] considered $K_{s,s}$-factors in bipartite graphs for all $s > 1$, the second author [24] gave the best possible minimum degree condition for this problem. Recently Hladký and Schacht [10] determined the minimum degree threshold for $K_{s,t}$-factors with $s < t$.

Let $G_r(N)$ denote the family of $r$-partite graphs with $N$ vertices in each of its partition sets. In an $r$-partite graph $G$, we use $\bar{\delta}(G)$ for the minimum degree from a vertex in one partition set to any other partition set. Fischer [8] proved almost perfect $K_3$-tilings in $G_3(N)$ with $\bar{\delta}(G) \geq 2N/3$ and Johansson [11] gives a $K_3$-factor with a less stringent degree condition $\bar{\delta}(G) \geq 2N/3 + O(\sqrt{N})$.

For general $r > 2$, Fischer [8] conjectured the following $r$-partite version of the Hajnal–Szemerédi Theorem: if $G \in G_r(N)$ satisfies $\bar{\delta}(G) \geq (r - 1)N/r$, then $G$ contains a $K_r$-factor. The first author and Szemerédi [20] proved this conjecture for $r = 4$. Csaba and Mydlarz [5] recently proved that the conclusion in Fischer’s conjecture holds if $\bar{\delta}(G) \geq k_r n$, where $k_r = r + O(\log r)$. On the other hand, Magyar and the first author [19] showed that Fischer’s conjecture is false for all odd $r \geq 3$: they constructed $r$-partite graphs $\Gamma(N) \in G_r(N)$ for infinitely many $N$ such that $\bar{\delta}(\Gamma(N)) = (r - 1)N/r$ and yet $\Gamma(N)$ contains no $K_r$-factor. Nevertheless, Magyar and the first author proved a theorem (Theorem 1.2 in [19]) which implies the following Corrádi-Hajnal-type theorem.

**Theorem 1.1 ([19])** If $G \in G_3(N)$ satisfies $\bar{\delta}(G) \geq (2/3)N + 1$, then $G$ contains a $K_3$-factor.

In this paper we extend this result to all $3$-colorable graphs. Our main result is on $K_{h,h,h}$-tiling.

**Theorem 1.2** For any positive real number $\gamma$ and any positive integer $h$, there is $N_0$ such that the following holds. Given an integer $N \geq N_0$ such that $N$ is divisible by $h$, if $G$ is a tripartite graph with $N$ vertices in each vertex class such that every vertex is adjacent to at least $(2/3 + \gamma)N$ vertices in each of the other classes, then then $G$ contains a $K_{h,h,h}$-factor.

Since the complete tripartite graph $K_{h,h,h}$ can be perfectly tiled by any $3$-colorable graph on $h$ vertices, we have the following corollary.

**Corollary 1.3** Let $H$ be a $3$-colorable graph of order $h$. For any $\gamma > 0$ there exists a positive integer $N_0$ such that if $N \geq N_0$ and $N$ divisible by $h$, then every $G \in G_3(N)$ with $\bar{\delta}(G) \geq (2/3 + \gamma)N$ contains an $H$-factor.

The Alon–Yuster theorem [2] says that for any $\gamma > 0$ and any $r$-colorable graph $H$ there exists $n_0$ such that every graph $G$ of order $n \geq n_0$ contains an $H$-factor if $n$ divisible by $h$ and $\delta(G) \geq (1 - 1/r)n + \gamma n$ (Komlós, Sárközy and Szemerédi [14] later reduced $\gamma n$ to a constant that depends only on $H$). Corollary 1.3 gives another proof of this theorem for $r = 3$ as follows. Let $G$ be a
graph of order \( n = 3N \) with \( \delta(G) \geq 2n/3 + 2\gamma n \). A random balanced partition of \( V(G) \) yields a subgraph \( G' \in \mathcal{G}_3(N) \) with \( \delta(G') \geq \delta(G)/3 - o(n) \geq (2/3 + \gamma)N \). We then apply Corollary 1.3 to \( G' \) obtaining an \( H \)-factor in \( G' \), hence in \( G \).

Instead of proving Theorem 1.2, we actually prove the stronger Theorem 1.4 below. Given \( \gamma > 0 \), we say that \( G = (V^{(1)}, V^{(2)}, V^{(3)}; E) \in \mathcal{G}_3(N) \) is in the extreme case with parameter \( \gamma \) there are three sets \( A_1, A_2, A_3 \) such that \( A_i \in V^{(i)} \), \( |A_i| = \lfloor N/3 \rfloor \) for all \( i \) and

\[
d(A_i, A_j) := d(A_i, A_j) = \frac{e(A_i, A_j)}{|A_i||A_j|} \leq \gamma
\]

for \( i \neq j \). If \( G \in \mathcal{G}_3(N) \) satisfies \( \delta(G) \geq (2/3 + \gamma)N \), then \( G \) is not in the extreme case with parameter \( \gamma \). In fact, any two sets \( A \) and \( B \) of size \( \lfloor N/3 \rfloor \) from two different vertex classes satisfy \( \text{deg}(a, B) \geq \gamma N \) and consequently \( d(A, B) > \gamma \). Theorem 1.2 thus follows from Theorem 1.4, which is even stronger because of its weaker assumption \( \delta(G) \geq (2/3 - \varepsilon)N \).

**Theorem 1.4** Given any positive integer \( h \) and any \( \gamma > 0 \), there exists an \( \varepsilon > 0 \) and an integer \( N_0 \) such that whenever \( N \geq N_0 \), and \( h \) divides \( N \), the following occurs: If \( G \in \mathcal{G}_3(N) \) satisfies \( \delta(G) \geq (2/3 - \varepsilon)N \), then either \( G \) contains a \( K_{h,h,h} \)-factor or \( G \) is in the extreme case with parameter \( \gamma \).

The following proposition shows that the minimum degree \( \delta(G) \geq (2/3 + \gamma)N \) in Theorem 1.2 can not be replaced by \( 2N/3 + h - 2 \).

**Proposition 1.5** Given any positive integer \( h \geq 2 \), there exists an integer \( q_0 \) such that for any \( q \geq q_0 \), there exists a tripartite graph \( G_0 \in \mathcal{G}_3(N) \) with \( N = 3qh \) such that \( \delta(G_0) = 2qh + (h - 2) \) and \( G_0 \) has no \( K_{h,h,h} \)-factor.

The structure of the paper is as follows. We first prove Proposition 1.5 in Section 2. After stating the Regularity Lemma and Blow-up Lemma in Section 3, we prove Theorem 1.4 in Section 4. We give concluding remarks and open problems in Section 5.

## 2 Proof of Proposition 1.5

In a tripartite graph \( G = (A, B, C; E) \), the graphs induced by \((A, B)\), \((A, C)\) and \((B, C)\) are called the natural bipartite subgraphs of \( G \). First we need to construct a balanced tripartite \( K_3 \)-free graph in which all natural bipartite graphs are regular and \( C_4 \)-free. Our construction below is based on the construction in [24] of sparse regular bipartite graphs with no \( C_4 \).

**Lemma 2.1** For each integer \( d \geq 0 \), there exists an \( n_0 \) such that, if \( n \geq n_0 \), there exists a balanced tripartite graph, \( Q(n, d) \) on \( 3n \) vertices such that each of the 3 natural bipartite subgraphs are \( d \)-regular with no \( C_4 \) and \( Q(n, d) \) has no \( K_3 \).
Proof. A Sidon set is a set of integers such that sums \( i + j \) are distinct for distinct pairs \( i, j \) from the set. Let \([n] = \{1, \ldots, n\}\). It is well known (e.g., [7]) that \([n] \) contains a Sidon set of size about \( \sqrt{n} \) for large \( n \). Suppose that \( n \) is sufficiently large. Let \( S \) be a \( d \)-element Sidon subset of \([\frac{n}{3}] - 1\).

Given two copies of \([n]\), \( A \) and \( B \), we construct a bipartite graph \( P(A, B) \) on \((A, B)\) whose edges are (ordered) pairs \( ab \), \( a \in A \), \( b \in B \) such that \( b - a \pmod{n} \in S \). It is shown in [24] that \( P(A, B) \) is \( d \)-regular with no \( C_4 \). Given three copies of \([n]\), \( A \), \( B \) and \( C \), let \( Q \) be the union of \( P(A, B) \), \( P(B, C) \) and \( P(C, A) \). In order to show that \( Q \) is the desired graph \( Q(n, d) \), we need to verify that \( Q \) is \( K_3 \)-free. In fact, if \( a \in A \), \( b \in B \), and \( c \in C \) form a \( K_3 \), then there exist \( i, j, k \in S \) such that

\[
\begin{align*}
    b &\equiv a + i, \quad c \equiv b + j, \quad a \equiv c + k \pmod{n},
\end{align*}
\]

which implies that \( i + j + k \equiv 0 \pmod{n} \). But this is impossible for \( i, j, k \in [\frac{n}{3}] - 1 \). \( \square \)

Proof of Proposition 1.5. We will construct 9 disjoint sets \( A_j^{(i)} \) with \( i, j \in \{1, 2, 3\} \). The union \( A_1^{(1)} \cup A_2^{(1)} \cup A_3^{(1)} \) defines the \( i \)-th vertex-class, while the triple \( (A_j^{(1)}, A_j^{(2)}, A_j^{(3)}) \) defines the \( j \)-th column.

Construct \( G_0 \) as follows: For \( i = 1, 2, 3 \), let \( |A_1^{(i)}| = qh - 1 \), \( |A_2^{(i)}| = qh \) and \( |A_3^{(i)}| = qh + 1 \). Let the graph in column 1 be \( Q(qh - 1, h - 3) \), the graph in column 2 be \( Q(qh, h - 2) \) and the graph in column 3 be \( Q(qh + 1, h - 1) \). If two vertices are in different columns and different vertex-classes, then they are also adjacent. It is easy to verify that \( \delta(G_0) = 2qh + (h - 2) \).

Suppose, by way of contradiction, that \( G_0 \) has a \( K_{h,h,h} \)-factor. Since there are no triangles and no \( C_4 \)'s in any column, the intersection of a copy of \( K_{h,h,h} \) with a column is either a star, with all leaves in the same vertex-class, or a set of vertices in the same vertex-class. So, each copy of \( K_{h,h,h} \) has at most \( h \) vertices in column 3. A \( K_{h,h,h} \)-factor has exactly \( 3q \) copies of \( K_{h,h,h} \) and so the factor has at most \( 3qh \) vertices in column 3. But there are \( 3qh + 3 \) vertices in column 3, a contradiction.

3 The Regularity Lemma and Blow-up Lemma

The Regularity Lemma and the Blow-up Lemma are main tools in the proof of Theorem 1.4. Let us first define \( \varepsilon \)-regularity and \((\varepsilon, \delta)\)-super-regularity.

Definition 3.1 Let \( \varepsilon > 0 \). Suppose that a graph \( G \) contains disjoint vertex-sets \( A \) and \( B \).

1. The pair \((A, B)\) is \( \varepsilon \)-regular if for every \( X \subseteq A \) and \( Y \subseteq B \), satisfying \( |X| > \varepsilon |A| \), \( |Y| > \varepsilon |B| \), we have \( |d(X, Y) - d(A, B)| < \varepsilon \).

2. The pair \((A, B)\) is \((\varepsilon, \delta)\)-super-regular if \((A, B)\) is \( \varepsilon \)-regular and \( \deg(a, B) > \delta |B| \) for all \( a \in A \) and \( \deg(b, A) > \delta |A| \) for all \( b \in B \).

The celebrated Regularity Lemma of Szemerédi [22] has a multipartite version (see survey papers [15, 16]), which guarantees that when applying the lemma to a multipartite graph, every resulting
cluster is from one partition set. Given a vertex \(v\) and a vertex set \(S\) in a graph \(G\), we define \(\deg(v, S)\) as the number of neighbors of \(v\) in \(S\).

**Lemma 3.2 (Regularity Lemma - Tripartite Version)** For every positive \(\varepsilon\) there is an \(M = M(\varepsilon)\) such that if \(G = (V, E)\) is any tripartite graph with partition sets \(V^{(1)}, V^{(2)}, V^{(3)}\) of size \(N\), and \(d \in [0, 1]\) is any real number, then there are partitions of \(V^{(i)}\) into clusters \(V^{(i)}_0, V^{(i)}_1, \ldots, V^{(i)}_k\) for \(i = 1, 2, 3\) and a subgraph \(G' = (V, E')\) with the following properties:

- \(k \leq M\),
- \(|V^{(i)}_0| \leq \varepsilon n\) for \(i = 1, 2, 3\),
- \(|V^{(i)}_j| = L \leq \varepsilon n\) for all \(i = 1, 2, 3\) and \(j \geq 1\),
- \(\deg_{G'}(v, V^{(i')}) > \deg_G(v, V^{(i)}) - (d + \varepsilon)N\) for all \(v \in V^{(i)}\) and \(i \neq i'\),
- all pairs \((V^{(i)}_j, V^{(i')}_{j'})\), \(i \neq i', 1 \leq j, j' \leq k,\) are \(\varepsilon\)-regular under \(G'\), each with density either 0 or exceeding \(d\).

We will also need the Blow-up Lemma of Komlós, Sárközy and Szemerédi [13].

**Lemma 3.3 (Blow-up Lemma)** Given a graph \(R\) of order \(r\) and positive parameters \(\delta, \Delta\), there exists an \(\varepsilon > 0\) such that the following holds: Let \(N\) be an arbitrary positive integer, and let us replace the vertices of \(R\) with pairwise disjoint \(N\)-sets \(V_1, V_2, \ldots, V_r\). We construct two graphs on the same vertex-set \(V = \bigcup V_i\). The graph \(R(N)\) is obtained by replacing all edges of \(R\) with copies of the complete bipartite graph \(K_{N,N}\) and a sparser graph \(G\) is constructed by replacing the edges of \(R\) with some \((\varepsilon, \delta)\)-super-regular pairs. If a graph \(H\) with maximum degree \(\Delta(H) \leq \Delta\) can be embedded into \(R(N)\), then it can be embedded into \(G\).

## 4 Proof of Theorem 1.4

In this section we prove Theorem 1.4. Let us first sketch our proof. We first apply the Regularity Lemma to \(G\) partitioning each vertex class into \(\ell\) clusters and an exceptional set. Next define the cluster graph \(G_r\) where clusters are adjacent if the pair is regular with positive density, which is 3-partite with \(\delta(G_r)\) is almost \(2\ell/3\). The so-called fuzzy tripartite theorem of [19] states that either \(G_r\) is in the extreme case (then \(G\) is in the extreme case) or \(G_r\) has a \(K_3\)-factor. Now suppose that \(G_r\) has a \(K_3\)-factor \(S = \{S_1, \ldots, S_\ell\}\). We first move a small amount of vertices from each cluster to the exceptional sets such that in each \(S_j\), all three pairs are super-regular and the three clusters have the same size, which is a multiple of \(h\). If we now apply the Blow-up Lemma to each \(S_j\), then we will obtain a \(K_{h,h,h}\)-factor covering all the non-exceptional vertices of \(G\). However, we need to
handle the exceptional sets first. We first remove some copies of $K_{h,h,h}$, each of which contains one vertex from the exceptional sets, and $2h - 1$ vertices from some cluster-triangle $S_j$. In order to make the sizes of all clusters divisible by $h$, we group all but a constant number of exceptional vertices into $h$-element sets such that all $h$ vertices in each set associate with the same $S_j$. We remove a few more copies of $K_{h,h,h}$ from some triangles in $G_r$ to balance the sizes of three clusters in each $S_j$. In order to handle the remaining constant number of exceptional vertices, we need to set up a structure in $G_r$ such that we can move a constant number of vertices from any $S_j$ to $S_1$. Since such a structure may contain triangles of $G_r$ that are not not in $S$, it must be set up before handing any exceptional vertex. At last, we apply the Blow-up Lemma to each $S_j$ to complete the $K_{h,h,h}$-factor of $G$. This ends the proof sketch.

Note that our proof follows the approach in [19], which has a different way of handling exceptional vertices from the bipartite case [24]. Although a $K_{h,h,h}$-tiling is more complex than a $K_3$-tiling, our proof is not longer than the non-extreme case in [19] because we take advantage of results from [19].

Let us now start the proof. We assume that $N$ is large, and without loss of generality, assume that $\gamma \ll \frac{1}{h}$. We find small constants $d_1, \varepsilon, \text{and } \varepsilon_1$ such that (actual dependencies result from Lemmas 4.1, 4.4, 4.7, and 3.3):

$$\varepsilon_1 \ll 2\varepsilon = d_1 \ll \gamma. \quad (1)$$

For simplicity, we will will refrain from using floor or ceiling functions when they are not crucial.

Begin with a tripartite graph $G = (V^{(1)}, V^{(2)}, V^{(3)}; E)$ with $|V^{(1)}| = |V^{(2)}| = |V^{(3)}| = N$ such that $\delta(G) \geq (2/3 - \varepsilon)N$. Apply the Regularity Lemma (Lemma 3.2) with $\varepsilon_1$ and $d_1$, partitioning each $V^{(i)}$ into $\ell$ clusters $V^{(i)}_1, \ldots, V^{(i)}_\ell$ of size $L \leq \varepsilon_1 N$ and an exceptional set $V^{(i)}_0$ of size at most $\varepsilon_1 N$. To distinguish from other clusters, we do not consider $V^{(i)}_0, i = 1, 2, 3$, as clusters, we call them exceptional sets instead. Later in the proof, the exceptional sets may grow in size, but will always remain of size $O(\varepsilon_1 N)$. We call the vertices in the exceptional sets exceptional vertices.

Let $G'$ be the subgraph of $G$ defined in the Regularity Lemma. We define the reduced graph (or cluster graph) $G_r$ to be 3-partite graph whose vertices are clusters $V_j^{(i)}$, $j \geq 1$, $i = 1, 2, 3$, and two clusters are adjacent if and only if they form an $\varepsilon_1$-regular pair of density at least $d_1$ in $G'$. We will use the same notation $V_j^{(i)}$ for a set in $G$ and a vertex in $G_r$. Let $U^{(1)}, U^{(2)}, U^{(3)}$ denote three partition sets of $G_r$. We know that $|U^{(i)}| = \ell$. We observe that $\delta(G_r) \geq (2/3 - 2d_1)\ell$. In fact, consider a cluster $C \in U^{(i)}$ and a vertex $x \in C$, the number $m$ of clusters in $U^{(i')}$ ($i' \neq i$) that are adjacent to $C$ satisfies

$$\left(\frac{2}{3} - \varepsilon\right)N - (d_1 + \varepsilon_1)N \leq \deg_G(v, V^{(i')}) - (d_1 + \varepsilon_1)N \leq \deg_{G_r}(x, V^{(i')}) \leq mL.$$ 

Since $N \geq L\ell$ and $\varepsilon + \varepsilon_1 \leq d_1$, we have $m \geq (2/3 - \varepsilon - d_1 - \varepsilon_1)\ell \geq (2/3 - 2d_1)\ell$.

Assume that $G$ is not in the extreme case with parameter $\gamma$. We claim that $G_r$ is not in the extreme case with parameter $\gamma/3$. Suppose instead, that there are subsets $S_i \subset U^{(i)}, i = 1, 2, 3$, of size $\ell/3$
with density at most $\gamma/3$. Let $A_i$ denote the set of all vertices of $G$ contained in a cluster of $S_i$. Then $N(1-\varepsilon)/3 \leq |A_i| = L\ell/3 \leq N/3$ because $L\ell \geq (1-\varepsilon)N$. The number of edges of $G$ between $A_i$ and $A_{i'}$, $i \neq i'$, is at most
\[
eq_G(A_i, A_{i'}) \leq e_G'(A_i, A_{i'}) + |A_i|(d_1 + \varepsilon_1)N \leq \frac{\gamma}{3} \left( \frac{\ell}{3} \right)^2 L^2 + (d_1 + \varepsilon_1) \frac{N^2}{3} \leq 2\varepsilon \left( \frac{N}{3} \right)^2,
\]
provided that $9(d_1 + \varepsilon_1) \leq \gamma$. After adding at most $\varepsilon N/3$ vertices to each $A_i$, we obtain three subsets of $V^{(1)}, V^{(2)}, V^{(3)}$ of size $N/3$ with pairwise density at most $(2\gamma/3 + \varepsilon_1) \leq \gamma$ in $G$.

**Step 1: Find a $K_3$-factor in $G_r$**

We apply the following result (Theorem 2.1 in [19]) to the reduced graph $G_r$ with $\alpha = \gamma/3$ and $\beta = 2d_1$.

**Lemma 4.1 (Fuzzy tripartite theorem [19])** For any $\alpha > 0$, there exist $\beta > 0$ and $\ell_0$, such that the follows holds for all $\ell > \ell_0$. Every balanced 3-partite graph $R \in G_3(\ell)$ with $\delta(R) \geq (2/3 - \beta)\ell$ either contains a $K_3$-factor or is in the extreme case with parameter $\alpha$.

Since $G_r$ is not in the extreme case with parameter $\gamma/3$, it must contain a $K_3$-factor $S = \{S_1, S_2, \ldots, S_\ell\}$. After relabeling, we assume that $S_j = \{V_j^{(1)}, V_j^{(2)}, V_j^{(3)}\}$ for all $j$. In $G_r$, we call these fixed triangles $S_1, \ldots, S_\ell$ columns and consider $U^{(1)}, U^{(2)}, U^{(3)}$ as rows.

**Step 2: Make pairs in $S_j$ super-regular**

For each $S_j$, remove a vertex $v$ from a cluster in $S_j$ and place it in the exceptional set if $v$ has fewer than $(d_1 - \varepsilon_1)L$ neighbors in one of the other clusters of $S_j$. By $\varepsilon_1$-regularity, there are at most $2\varepsilon_1L$ such vertices in each cluster. Remove more vertices if necessary to ensure that each non-exceptional cluster is of the same size and the size is divisible by $h$.

Slicing Lemma is a well-known fact that regularity is maintained when small modifications are made to the clusters:

**Proposition 4.2 (Slicing Lemma, Fact 1.5 in [19])** Let $(A, B)$ be an $\varepsilon$-regular pair with density $d$, and, for some $\alpha > \varepsilon$, let $A' \subset A$, $|A'| \geq \alpha|A|$, $B' \subset B$, $|B'| \geq \alpha|B|$. Then $(A', B')$ is an $\varepsilon'$-regular pair with $\varepsilon' = \max\{\varepsilon/\alpha, 2\varepsilon\}$, and for its density $d'$, we have $|d' - d| < \varepsilon$.

Applying Proposition 4.2 with $\alpha = 1 - 2\varepsilon_1$, any pair of clusters which was $\varepsilon_1$-regular with density at least $d_1$ is now $(2\varepsilon_1)$-regular with density at least $d_1 - \varepsilon_1$ (because $\varepsilon_1 < 1/4$). Furthermore, each pair in the cluster-triangles $S_j$ is $(2\varepsilon_1, d_1 - 3\varepsilon_1)$-super-regular. Each of the three exceptional sets are now of size at most $\varepsilon_1N + \ell(2\varepsilon_1L) \leq 3\varepsilon_1N$. 


Figure 1: An illustration of how cluster $V_1^{(1)}$ is reachable from a cluster $C$.

**Remark:** Because all the pairs in $S_j$ are super-regular and the complete tripartite graph on $(V^{(1)}_i, V^{(2)}_i, V^{(3)}_i)$ contains a $K_{h,h,h}$-factor, the Blow-up Lemma says that $S_j$ also contains $K_{h,h,h}$-factor.

**Step 3: Create red copies of $K_{h,h,h}$**

In this step, we link each cluster to the one in $S_1$ from the same partition class by creating some triangles in $G_r$. Its purpose is to be able to handle a small discrepancy of sizes among the three clusters that comprise $S_j$ in Step 5.

**Definition 4.3** In a tripartite graph $R = (U^{(1)}, U^{(2)}, U^{(3)}; E)$, one vertex $x \in U^{(1)}$ (the cases of $x \in U^{(2)}$ or $U^{(3)}$ are defined accordingly) is reachable from another vertex $y \in U^{(1)}$ in $R$, if there is a chain of triangles $T_1, \ldots, T_{2k}$ with $T_j = \{T_j^{(1)}, T_j^{(2)}, T_j^{(3)}\}$ and $T_j^{(i)} \in U^{(i)}$ for $i = 1, 2, 3$ such that the following occurs:

1. $x = T_1^{(1)}$ and $y = T_{2k}^{(1)}$,
2. $T_{2j-1}^{(2)} = T_{2j}^{(2)}$ and $T_{2j-1}^{(3)} = T_{2j}^{(3)}$, for $j = 1, \ldots, k$, and
3. $T_{2j}^{(1)} = T_{2j+1}^{(1)}$, for $j = 1, \ldots, k - 1$.

Figure 1 illustrates that $V_1^{(1)}$ is reachable from another cluster $C$ by using four triangles. The Reachability Lemma (Lemma 2.6 in [19]) says that every cluster of $S_1$ is reachable from any other cluster in the same class by using at most four triangles in $G_r$. Note that these triangles are not necessarily the fixed triangles $S_j$. The statement of the Reachability Lemma in [19] refers to the reduced graph, but its proof, in fact, proves the following general statement:

**Lemma 4.4 (Reachability Lemma)** For any $\alpha > 0$, there exist $\beta > 0$ and $\ell_0$, such that the following holds for all $\ell \geq \ell_0$. Let $R \in \mathcal{G}_3(\ell)$ be a balanced 3-partite graph with $\tilde{\delta}(R) \geq (2/3 - \beta)\ell$. Then either each vertex is reachable from every other vertex in the same class by using at most four triangles or $R$ is in the extreme case with parameter $\alpha$.  

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Let $C \neq V_1^{(1)}$ be a cluster in $U^{(1)}$ and let $T_1, T_2$ or $T_1, T_2, T_3, T_4$ be cluster-triangles used when $V_1^{(1)}$ is reachable from $C$. Note that $T_1 \cap U^{(1)} = S_1^{(1)}$ and either both $k = 1$ and $T_2 \cap U^{(1)} = C$ or $k = 2$, $T_2 \cap U^{(1)} = T_3 \cap U^{(1)} = C'$ and $T_4 \cap U^{(1)} = C$.

We need a special case of a well-known embedding lemma in [15], which says that three reasonably large subsets of three clusters that form a triangle induce a copy of $K_{h,h,h}$.

**Proposition 4.5 (Key Lemma, Theorem 2.1 in [15])** Let $\varepsilon, d$ be positive real numbers and $h, L$ be positive integers such that $(d - \varepsilon)^2h > \varepsilon$ and $\varepsilon(d - \varepsilon)L \geq h$. Suppose that $X_1, X_2, X_3$ are clusters of size $L$ and any pair of them is $\varepsilon$-regular with density at least $d$. Let $A_i \subseteq X_i$, $i = 1, 2, 3$ be three subsets of size at least $(d - \varepsilon)L$. Then $(A_1, A_2, A_3)$ contains a copy of $K_{h,h,h}$.

If $k = 1$, then we pick a vertex $v \in C$ and apply Proposition 4.5 to find a copy of $K_{h,h,h}$, called $H'$, in the cluster triangle $T_1$ such that $H' \cap V^{(2)}$ and $H' \cap V^{(3)}$ are in the neighborhood of $v$. If $k = 2$, then we first pick a vertex $v \in C$ and apply Proposition 4.5 to find a copy of $K_{h,h,h}$, called $H''$, in the cluster triangle $T_3$ such that $H'' \cap V^{(2)}$ and $H'' \cap V^{(3)}$ are in the neighborhood of $v$. Next we pick a vertex $v' \in H'' \cap V^{(1)}$ (call it special) and apply Proposition 4.5 to find a copy of $K_{h,h,h}$, called $H'$, in the cluster triangle $T_1$ such that $H' \cap V^{(2)}$ and $H' \cap V^{(3)}$ are in the neighborhood of $v'$.

Color all of the vertices in $H'$ and in $H''$ (if it exists) red and the vertex in $C$ orange. Note that the special vertex in $H''$ (if exists) is colored red. If a vertex is not colored, we will heretofore call it uncolored. Repeat this $5h$ times for each cluster not in $S_1$. In this process all but a constant number of vertices in each cluster remain uncolored since $h$ is a constant and $G_r$ consists of a constant number (that is, $3\ell$) of clusters. This is why we can repeatedly apply Proposition 4.5 ensuring that all the red copies of $K_{h,h,h}$ and orange vertices are vertex-disjoint.

At the end, each cluster not in $S_1$ has $5h$ orange vertices (the clusters in $S_1$ have no orange vertex). Each cluster has at most $3(\ell - 1)(5h)(h) < 15th^2$ red vertices because there are $3(\ell - 1)$ clusters not in $S_1$, the process is iterated $5h$ times for each of them and a cluster gets at most $h$ vertices colored red with each iteration.

**Remark:** This preprocessing ensures that we may later transfer at most $5h$ vertices from any cluster $C$ to $S_1$ in the following sense: Without loss of generality, suppose $C$ is a cluster in $V^{(1)}$. In the case when $k = 2$ (see Figure 1), identify an orange vertex $v \in C$ and its corresponding red subgraphs $H'$ and $H''$, including the special vertex $v' \in C'$. (The case where $k = 1$ is similar but simpler.) Recolor $v$ red and uncolor a vertex $u \in H' \cap V_1^{(1)}$. The red vertices still form two copies of $K_{h,h,h}$, one is $H' - \{u\} + \{v'\}$, and the other one is $H'' - \{v'\} + \{v\}$. The number of non-red vertices is decreased by one in $C$ but is increased by one in $V_1^{(1)}$. We will do this in Step 5.

We now move some uncolored vertices from clusters to the corresponding exceptional set such that the three clusters in the same column (some $S_j$) have the same number of uncolored vertices. In other words, three clusters in any $S_j$ are balanced in terms of uncolored vertices. (Note that
this number is always divisible by \( h \) because the numbers of red vertices and orange vertices are divisible by \( h \).) Thus, at most \( 15\ell h^2 \) vertices can be removed from a cluster. The three exceptional sets have the same size, at most \( 3\varepsilon_1 N + 15\ell^2 h^2 \leq 4\varepsilon_1 N \). Each cluster still has at least \((1 - 2\varepsilon_1) L - 15\ell h^2 > (1 - 3\varepsilon_1) L \) uncolored vertices.

**Step 4: Reduce the sizes of exceptional sets**

At present the exceptional sets \( V_{0}^{(i)}, i = 1, 2, 3 \), are all of the same size, which is at most \( 4\varepsilon_1 N \) and divisible by \( h \). Suppose this size is at least \( 6h \). We will remove some copies of \( K_{h,h,h} \) from \( G \) such that \( |V_{0}^{(i)}| \leq 5h \) eventually.

First, we say a vertex \( v \in V_{0}^{(i)} \) belongs to a cluster \( V_j^{(i)} \) if \( \deg(v, V_j^{(i)}) \geq d_1 L \) for all \( i' \neq i \). Using the minimum-degree condition, for fixed \( i' \neq i \), the number of clusters \( V_j^{(i')} \) such that \( \deg(v, V_j^{(i')}) < d_1 L \) is at most

\[
\frac{(1/3 + \varepsilon)N}{(1 - 3\varepsilon_1) L - d_1 L} \leq \frac{(1/3 + \varepsilon)\ell}{(1 - 3\varepsilon_1 - d_1)(1 - \varepsilon_1)}.
\]

Using (1), the expression in (2) is at most \((1/3 + d_1)\ell \). Thus, \( v \) is adjacent to at least \( d_1 L \) uncolored vertices in at least \((2/3 - d_1)\ell \) clusters in \( V^{(i')} \) for some \( i' \neq i \). Hence, each vertex in \( V_{0}^{(i)} \) belongs to at least \((1/3 - 2d_1)\ell \) clusters.

If a vertex \( v \in V_{0}^{(i)} \) belongs to a cluster \( V_j^{(i)} \), then we may insert \( v \) into \( V_j^{(i)} \) (or loosely speaking, insert \( v \) into \( S_j \)) in the following sense. We permanently remove a copy of \( K_{h,h,h} \) from \( G \) which consists of \( v \), \( h - 1 \) vertices from \( V_j^{(i)} \) and \( h \) vertices from each of \( V_j^{(k)}, k \neq i \). Proposition 4.5 guarantees the existence of this \( K_{h,h,h} \).

In order to maintain the size of each cluster as a multiple of \( h \), we will bundle exceptional vertices into \( h \)-element sets and handle all \( h \) vertices from an \( h \)-element set at a time as follows.

**Claim 4.6** Given a subset \( Y \subseteq V_{0}^{(i)} \) of at least \( 3h \) vertices and a subset \( U' \subseteq U^{(i)} \) of at least \((1 - d_1)\ell \) clusters, there are \( h \) vertices of \( Y \) that belong to the same cluster \( C \) from \( U' \).

**Proof.** Suppose instead, that at most \( h - 1 \) vertices of \( Y \) belongs to each cluster \( C \in U' \). From earlier calculations and the assumption \( |U'| \geq (1 - d_1)\ell \), we know that each vertex of \( Y \) belongs to at least \((1/3 - 3d_1)\ell \) clusters. By double counting the number of pairs \((v, C)\) such that \( v \in Y \) belongs to a cluster \( C \in U' \), we have

\[
3h \left( \frac{1}{3} - 3d_1 \right) \ell \leq (h - 1)\ell,
\]

which implies that \( 9hd_1 \geq 1 \), contradicting \( d_1 \ll 1 \). \( \square \)

Starting from \( Y = V_{0}^{(i)} \) and \( U' = U^{(i)} \), we apply Claim 4.6 four times to find four disjoint \( h \)-element subsets \( W_{1}^{(i)}, \ldots, W_{4}^{(i)} \) of \( V_{0}^{(i)} \) whose vertices belong to clusters \( C_{1}^{(i)}, \ldots, C_{4}^{(i)} \), respectively.
The reason why we need four $h$-element sets can be seen below when we apply Lemma 4.7. We can ensure that $C_1^{(i)}, \ldots, C_4^{(i)}$ are different by letting $U' = U^{(i)} \setminus \{C_{j'}^{(i)} : j' < j\}$ when we select $C_j^{(i)}$.

We now insert $W_j^{(i)}$ into $C_j^{(i)}$ for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$ by removing total $12h$ copies of $K_{h,h,h}$. All of these copies of $K_{h,h,h}$ are removed permanently, they will be a part of the final $K_{h,h,h}$-factor of $G$. As a result, each $C_j^{(i)}$ has $h$ more vertices than the other two clusters in the same column (unless accidentally more than one $C_j^{(i)}$ fall into the same column).

The Almost-covering Lemma (Lemma 2.2 in [19]) can help us to balance the sizes of each column:

**Lemma 4.7 (Almost-covering Lemma [19])** For any $\alpha > 0$, there exist $\beta > 0$ and $m_0$, such that the following holds for all $m \geq m_0$. Let $R \in G_3(m)$ be a balanced 3-partite graph with $\delta(R) \geq (2/3 - \beta)m$. Suppose that $T_0$ is a partial $K_3$-tiling in $R$ with $|T| < m - 3$. Then, either

1. there exists a partial $K_3$-tiling $T$ with $|T'| > |T|$ but $|T' \setminus T| \leq 15$, or

2. $R$ is in the extreme case with the parameter most $\alpha$.

Let $\tilde{G}$ be a new 3-partite graph obtained from adding four new vertices to each vertex class of $G_r$. The new 12 vertices are clones of the clusters $C_j^{(i)}$ for $i = 1, 2, 3$, $j = 1, 2, 3, 4$, and we denote them by $\tilde{C}_j^{(i)}$. The clones have the same adjacency in $G_r$ as their originals. Let $m = \ell + 4$ be the size of vertex classes in $\tilde{G}$. We have $\tilde{\delta}(\tilde{G}) \geq (1/3 - 3d_1)m$ following from $\tilde{\delta}(G_r) \geq (1/3 - 2d_1)\ell$.

We apply Lemma 4.7 to $\tilde{G}$ with $\alpha = \gamma/3$, $\beta = 3d_1$, and $T = \{S_1, \ldots, S_\ell\}$ (then $|T| < m - 3$). The new graph $\tilde{G}$ is almost the same as $G_r$, provided $\ell$ is large enough, which we guaranteed when we applied the Regularity Lemma. Thus, $\tilde{G}$ is not in the extreme case (otherwise $G_r$ is in the extreme case). Lemma 4.7 thus provides a larger partial triangle-cover $T'$ with $|T' \setminus T| \leq 15$. For each triangle $T \in T' \setminus T$, we permanently remove a copy of $K_{h,h,h}$ from the uncolored vertices of $T$. For each cluster $C$ that is not covered by the larger $T'$, take an arbitrary set of $h$ uncolored vertices from $C$ and place it into the exceptional set. As result, all the clusters covered by $T' \cap T$ experience no changes while all other clusters lose $h$ uncolored vertices; therefore the three clusters in each $S_j$ remain balanced. The net change in each $V_0^{(i)}$ is the same for all $i$ and each loses at least $h$ vertices because $|T'| > |T|$.

We repeat the process of creating $W_j^{(i)}, C_j^{(i)}, \tilde{G}$, and enlarging $T = \{S_1, \ldots, S_\ell\}$ in $\tilde{G}$ by Lemma 4.7 until the number of vertices remaining in each exceptional set is less than $6h$. There is one caveat: If too many vertices are removed from the clusters of $S_j$, then we will not be able to apply the Blow-up Lemma later. If in the entire process, at least $d_1L/3$ (uncolored) vertices are removed from a cluster $C$ of $S_j$, then both $C$ and $S_j$ are called dead (otherwise live).

The number of dead cluster-triangles is not very large. To see this, there are three ways for vertices to leave a cluster. First, they are placed in a $K_{h,h,h}$ with a vertex from the exceptional set, so each vertex class $V^{(i)}$ loses at most $\sum_{i=1}^3 |V_0^{(i)}|h$ vertices in this way. Second, each time when we
apply Lemma 4.7, there are at most 15 triangles in $T' \setminus T$ and there are a total of $15h$ vertices lost to 15 copies of $K_{h,h,h}$. Third, there are at most 3 clusters not covered by $T'$ and they could lose $3h$ vertices to the exceptional sets. Since we apply Lemma 4.7 at most $|V_0^{(i)}|/h$ times, the total number of vertices that leave clusters is at most

$$3|V_0^{(i)}|/h + \left(|V_0^{(i)}|/h\right)(15h + 3h) = |V_0^{(i)}|(3h + 18) \leq 4\epsilon_1N(3h + 18).$$

The number of dead cluster triangles is at most

$$\frac{4\epsilon_1N(3h + 18)}{(d_1/3)L} \leq \frac{36(h + 6)\epsilon_1}{d_1(1 - \epsilon_1)} \ell < \frac{d_1}{2} \ell,$$

because $\epsilon_1 < d_1$.

Considering only live clusters has little impact on our earlier arguments. Each vertex in $V_0^{(i)}$ belongs to at least $(1/3 - 3d_1)\ell$ live clusters. By letting $U'$ be the set of available live clusters, we still have $|U'| \geq (1 - d_1)\ell$ when applying claim 4.6. After removing the edges incident with dead $S_j$’s, the minimum-degree condition in $\tilde{G}$ is still $\delta(\tilde{G}) \geq (2/3 - 3d_1)m$ and Lemma 4.7 can still be applied.

At the end each cluster (live or dead) has at least $(1 - 3\epsilon_1)L - d_1L/3$ uncolored vertices. Each of the three clusters in any $S_j$ has the same number of uncolored vertices, and this number is always divisible by $h$.

**Step 5: Insert the remaining exceptional vertices and apply the Blow-up Lemma**

At this stage, the exceptional sets $V_0^{(i)}$, $i = 1, 2, 3$ are all of the same size, divisible by $h$ and at most $5h$ (because it is less than $6h$). Consider a vertex $x \in V_0^{(1)}$ and insert $x$ into a live cluster $V_j^{(1)}$ to which $x$ belongs (as shown in Step 4, $x$ belongs to at least $(1/3 - 3d_1)\ell$ live clusters). As a result, $V_j^{(1)}$ loses $h - 1$ vertices while $V_j^{(2)}$ and $V_j^{(3)}$ each loses $h$ vertices. To balance $S_j$, we move a vertex from $V_j^{(1)}$ to $V_1^{(1)}$ following the remark in Step 3. As a result, $V_j^{(1)}$ loses one orange vertex, and $V_1^{(1)}$ gains an extra uncolored vertex. Repeat this to all the vertices in $V_0^{(1)} \cup V_0^{(2)} \cup V_0^{(3)}$. All $S_j$, $j > 1$, have the same number of non-red vertices among its three clusters. The same holds for $S_1$ because $|V_0^{(1)}| = |V_0^{(2)}| = |V_0^{(3)}|$. In addition, the number of non-red vertices in each cluster is at least $(1 - d_1/2)L$, and always a multiple of $h$.

Then, uncolor all the remaining orange vertices and remove all red copies of $K_{h,h,h}$ from $G$. Since each cluster now has at least $(1 - d_1/2)L$ vertices, by the Slicing Lemma, any pair of clusters in $S_j$ is $(\epsilon_1/2)$-regular. Furthermore, each vertex in one cluster of $S_j$ is adjacent to at least $(d_1 - \epsilon_1)L - d_1L/2$ vertices in any other cluster of $S_j$. Hence all pairs in $S_j$ are $(\epsilon_1/2, d_1/3)$-super-regular. We finally apply the Blow-up Lemma to each $S_j$ to complete the $K_{h,h,h}$-factor of $G$. 
5 Concluding Remarks

- We could reduce the error term $\gamma N$ in Theorem 1.2 to a constant $C = C(h)$ by showing that if $G \in \mathcal{G}_3(N)$ is in the extreme case with sufficiently small $\gamma$ and $\bar{\delta}(G) \geq 2N/3 + C$, then $G$ contains a $K_{h,h,h}$-factor. Unfortunately, the methods involve a detailed case analysis which is too long to be included in this paper. However, we can summarize them as follows. Given a positive integer $h$, let $f(h)$ be the smallest $m$ for which there exists an $N_0$ such that every balanced tripartite graph $G \in \mathcal{G}_3(N)$ with $N \geq N_0$, $h$ divides $N$, and $\bar{\delta}(G) \geq m$ contains a $K_{h,h,h}$-factor. Suppose that $N = (6q + r)h$ with $0 \leq r \leq 5$. Then, from Proposition 1.5 and that unpublished work:

$$f(h) = \begin{cases} \frac{2N}{3} + h - 1, & \text{if } r = 0; \\ h \left[ \frac{2N}{3} \right] + h - 2 & \leq f(h) \leq h \left[ \frac{2N}{3} \right] + h - 1, & \text{if } r = 1, 2, 4, 5; \\ \frac{2N}{3} + h - 1 & \leq f(h) \leq \frac{2N}{3} + 2h - 1, & \text{if } r = 3. \end{cases}$$

We have no conjecture as to whether the upper or lower bound is correct.

- The task of obtaining a tight minimum pairwise degree condition for $K_r$-factors in $\mathcal{G}_r(N)$ becomes more challenging for larger $r$. The $r = 2$ case is very easy – we either consider a maximum matching or apply the König-Hall theorem. The $r = 3, 4$ cases become hard – [19] and [20] both applied the Regularity Lemma. At present a tight Hajnál–Szemerédi-type result is out of reach (though an approximate version was given by Csaba and Mydlarz [5]).

- We believe one can prove a similar result as Theorem 1.2 for tiling 4-colorable graphs in 4-partite graphs by adopting the approach of [20] and the techniques in this paper. In general, suppose that we know that every $r$-partite graph $G \in \mathcal{G}_r(n)$ with $\bar{\delta}(G) \geq cn$ contains a $K_r$-factor. Then applying the Regularity Lemma, one can easily prove that for any $\varepsilon > 0$ and any $r$-colorable $H$, every $G \in \mathcal{G}_r(n)$ with $\bar{\delta}(G) \geq (c + \varepsilon)n$ contains an $H$-tiling that covers all but $\varepsilon n$ vertices (this is similar to an early result of Alon and Yuster [1]). However, it is not clear how to reduce the number of leftover vertices to a constant, or zero (to get an $H$-factor).

As seen from this paper, we not only need a result on $K_r$-tiling but also the proof of such a result.

- Theorem 1.2 gives a near tight minimum degree condition $\bar{\delta} \geq (2/3 + o(1))N$ for tiling $K_{h,h,h}$. However, the coefficient $2/3$ may not be best possible for other 3-colorable graphs, e.g., $K_{1,2,3}$. In fact, when tiling a general (instead of 3-partite) graph with certain 3-colorable $H$, the minimum degree threshold given by Kühn and Osthus [17] has coefficient $1 - 1/\chi_{cr}(H)$ instead of $2/3$, where $\chi_{cr}(H)$ is the so-called critical chromatic number. It would be interesting to see if something similar holds for tripartite tiling.
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