In Memory of Michael O. Albertson, 1946 - 2009:
a collection of his outstanding conjectures and questions in graph theory.
June 19, 2009

Michael Albertson died in March after a short, intense battle with a rare form of thyroid cancer. He was a mathematician fully engaged in teaching, graph theory research, and the SIAM Discrete Mathematics (DM) community. He taught at Smith College for 37 years, where he was the L. Clark Seelye Professor. He produced research and scholarly work for 40 years and collaborated with many, including undergraduate students. A summary of his collaborative work can be found in a recent AWM Newsletter [H]. A complete list of his publications can be found at `math.smith.edu/~albertson`; copies of almost all his papers are linked to the web site and freely available.

Mike was chair of the SIAM DM Activity Group between 2000 and 2002, served on the SIAM Coordinating Committee for the annual AMS-MAA-SIAM Joint Math Meetings from 2006 to 2008, and was an enthusiastic supporter of the alternate-year SIAM DM Conferences, organizing many minisymposia for these. With K. Collins and R. Haas he ran the Combinatorists of New England (CoNE) weekend conferences, held several times a year at Smith College between 1993 and 2001.

Next March 26 - 28, 2010, at Smith College, a conference in memory of Mike will be held and run in the style of the CoNE conferences with a limited number of speakers, no parallel sessions, and plenty of time for discussion between talks. For more information on "CoNE Revisited: a celebration of the inspirations of Michael Albertson," see `www.math.smith.edu/cone/-MikeAlbertsonConference.html.`

In this article we pay tribute to Mike's research by presenting some of his many outstanding conjectures and questions; most are in the areas of chromatic, topological, geometric, and group-theoretic graph theory. All graph theory terminology can be found in [W, MT]; n always denotes the number of vertices of a graph.

Mike earned his Ph.D. in 1971 at the University of Pennsylvania, under the direction of Professor H.S. Wilf. We, myself included, were fortunate to study at Penn in the late 60s since Prof. Wilf had just returned from a 1966 sabbatical at Imperial College, London, and taught a course in the new area of Combinatorial Analysis and Graph Theory. In London Prof. Wilf found in *Math. Reviews,* and read, every paper in this new field, then began writing pioneering, now fundamental, papers in combinatorics. Since Mike wanted to work in an important area, he chose to work on aspects of the Four-Color Problem. Among other results, he worked on the Erdős-Vizing Conjecture [V; see Ber], which conjectures that every planar graph (on n vertices) contains an independent set of at least n/4 vertices. Mike established the existence of an independent set with at least 2n/9 vertices [A], and his concisely proven bound remains the best result today on this conjecture that is independent of the (now) Four Color Theorem [ApH, RSST].

David Berman, now Professor Emeritus of the University of New Orleans, also studied combinatorics at Penn with us, and became a lifelong collaborator with Mike. He and Mike conjectured [ABe2] a stronger result, implying Erdős-Vizing, that every planar graph contains an induced forest on at least n/2 vertices. They worked on Grünbaum's Conjecture [G] and showed that the acyclic chromatic number of the plane is at most 7 [ABe]. That conjecture was proven by

Borodin [Bor], who showed the best-possible bound to be 5, which also implies an induced forest on at least 2n/5 vertices.

Mike joined the Smith mathematics faculty in 1973 and I was hired there in 1976. Starting in 1973 we began work together on coloring and embedding problems in topological graph theory. We introduced the concept of *width* of an embedded graph, which is the length of the graph's shortest noncontractible cycle [AHu1], and found that concept useful, even essential, in proving topological results, such as the fact that every triangulation of a nonplanar surface contains a noncontractible cycle of length at most $\sqrt{2n}$ [AHu2]. As a consequence, all but at most $\sqrt{2n}$ vertices of a triangulation of the torus can 4-colored, using the Four Color Theorem. Mike recently reported that his most favorite conjecture, among all that he made, was that all but at most three vertices of a triangulation of the torus can be 4-colored [A2]. He also conjectured that for each nonplanar surface S, there is an integer f(S) such that all but at most f(S) vertices of a triangulation of S can be 4-colored. In [JT] these are known as the Albertson Four-Color Problem.

The concept of width is now a fundamental one in topological graph theory; it merits, for example, a whole chapter in [MT]. In 1982 Mike and Walter Stromquist [AS] introduced the idea of a *locally planar* embedded graph, which is one embedded with large width, where large typically is a function of the genus of the embedding surface. They proved that width at least 8 implies that a graph on the torus is 5-colorable and conjectured that locally planar graphs on each surface can be 5-colored. The latter conjecture was proven by Carsten Thomassen [T]. A more extensive summary of Mike's work on coloring graphs on surfaces can be found in [H2].

Mike went on to work on many generalizations of graph coloring, often with undergraduate students, including Karen Collins, and with colleague Ruth Haas. They obtained results on graph homomorphisms along with other chromatic-related problems. Mike worked with Doug West on the circular chromatic number, with Bojan Mohar on coloring vertices and faces of embedded graphs, on edge-coloring and list-coloring with R. Haas. If G is an r-chromatic graph and $t \le r$, let u(t, G) denote the maximum number of vertices in an induced t-colorable subgraph of G. Then in [AC] Mike and K. Collins showed that if f is a homomorphism from G to H and H is vertex-transitive, then for all t, $u(t,G) \ge u(t, H)$. Ruth Haas reports that her favorite conjecture with Mike and S. Grossman is the following [AGH]. Recall that if G is r-colorable, then for all $t \le r$, $u(t, G) \ge (t/r)n$. They conjecture that if G is s-list-colorable (also known as *s-choosable*) and if every vertex of G is assigned a list of t colors, with $t \le s$, then at least (t/s)n vertices of G can be t-list-colored. Apparently their related question is still open: if G is planar, can at least n/2 vertices of G be 2-list colored?

Mike worked with a number of people on pre-coloring extensions; I recommend highly his paper, "You can't paint yourself into a corner" [A3] for particularly good reading and a number of intriguing open questions.

In [AC2] Mike and K. Collins introduced the concept of the *distinguishing number, D(G),* of a graph G, which is the smallest number of labels needed, with labels attached to the vertices of G, so that no nontrivial automorphism of G preserves all the vertex labels. A number of papers have been and are being written in this area, including some by Mike and Debra Boutin that have a geometric flavor. A *geometric graph* is a straight-line drawing of a graph in the plane with vertices in general position, and its *geometric automorphisms* are those automorphisms that preserve both crossing edges and noncrossing edges. In [ABo] the authors studied the distinguishing number of geometric graphs using geometric automorphisms as follows: The *geometric distinguishing number* of a graph G is given by

$D_g(G) = \max\{D(\overline{G}): \overline{G}$ is a geometric realization of G$\}$.

They showed that for $3 \le n \le 6$, $D_g(K_n) = 3$ except when K_4 has a nonconvex drawing, and for that geometric graph $D(\overline{K_4}) = 4$. Then they ask whether for $n \ge 7$, $D_g(K_n) = 2$, or if not, whether there is an $N > 7$ such that for all $n \ge N$, $D_g(K_n) = 2$. This pattern of distinguishing number being initially 3, then decreasing to 2, is seen with cycles and with Kneser graphs [ABo2].

This study of geometric graphs leads one to think about the notoriously difficult area of crossing numbers. [A4] is a stimulating paper of Mike's that includes a mixture of coloring, independent sets, and geometric graphs. Given a geometric graph, two crossings are *dependent* if two edges of the crossings are incident with the same vertex, and otherwise they are *independent*. Mike conjectured: If G has a representation as a geometric graph with all crossings independent, then the chromatic number of G, $\chi(G)$, is at most 5. He proved $\chi(G) \le 6$, and I am pleased to report that this conjecture has just been proven by Daniel Král and Ladislav Stacho [KS].

At times Mike used the moniker "graphcolormike". I would propose also "planargraphmike" since from the beginning and throughout his career he was interested in planar and planar-like graphs. In 2007 at the 6th Slovenian International Conference on Graph Theory, Bled' 07, he gave a talk on "Independence ratios of nearly planar graphs." Nearly planar graphs include those of thickness 2; Mike considered thickness an especially challenging parameter. Though the chromatic number of thickness-2 graphs is known only to lie between 9 and 12, inclusive, Mike had asked in the mid 1990s (personal communication) whether every thickness-2 graph had $\alpha(G)/n \ge 1/8$, where α (G) is the independence number of G. A counterexample with $\alpha(G)/n = 2/17$ was found and given in [GS], and in Bled he asked more cautiously about how close $\alpha(G)/n$ might get to 1/9.

May we all continue to work on these intriguing, important conjectures and questions of Mike Albertson, to achieve success with these, and to remember him with these living testimonials to his mathematics and life.

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