

Ryan Martin

Extended Research Statement

The focus of my research is extremal graph theory and random combinatorial structures. I have also worked in a variety of other areas, including intersecting hypergraphs, the theory of positional games and Ramsey theory.

I use a number of tools in my research, notably probabilistic methods and, most prominently, Szemerédi's regularity lemma. I have used these and other techniques to address questions related to graphs, hypergraphs and combinatorial structures.

The following is a summary of some major facets of my research. I cannot cover all of the papers adequately, but I hope to summarize some of the major themes. Reference numbers correspond to the numbering of the papers in my CV and the reference numbers are hyperlinks to a preprint of the referenced manuscript.

Edit distance in graphs.

The edit distance of graphs is a question that relates very closely to other, very deep questions in extremal graph theory with some direct applications. The *edit distance* between two labeled graphs G and G' , $\text{Dist}(G, G')$ is the symmetric difference of the edge sets. The distance of a graph G from a property \mathcal{H} is $\text{Dist}(G, \mathcal{H}) \stackrel{\text{def}}{=} \min\{\text{Dist}(G, G') : G' \in \mathcal{H}\}$. The properties of primary concern in this area are hereditary properties. A *hereditary property* of graphs is one that is invariant under the deletion of vertices. That is, \mathcal{H} is a hereditary property if $H \in \mathcal{H}$ implies $H - v \in \mathcal{H}$ for all $v \in V(H)$.

Hereditary properties have been studied by a number of authors, notably Prömel and Steger, Scheinerman and Zito as well as Bollobás and Thomason. Edit distance for hereditary properties was first studied in [20], written with Maria Axenovich and André Kézdy where the properties in question are of the form $\text{Forb}(H)$; i.e, graphs with no induced copy of graph H . As in many of the previous results on hereditary properties, the key technique is to use Szemerédi's regularity lemma twice: once to a graph itself and once to each of the clusters that result from the first application.

A primary result of [20] is that, for any H ,

$$d^*(H) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \max_{|V(G)|=n} \{\text{Dist}(G, \text{Forb}(H))\} / \binom{n}{2}$$

exists and is bounded by two computable functions of H which differ by, at most, a multiplicative factor of two. In addition, $d^*(H)$ is determined exactly for a class of graphs H which includes self-complementary graphs.

The original motivation for the edit distance question was in the area of consensus trees. Two consensus trees are comparable if there is no induced path on 5 vertices in some corresponding bipartite graph. Biologists are interested in finding the maximum number of edit operations necessary to have no induced copy of such a path. In [11], written with

Axenovich, we answer this question and generalize it, asymptotically settling the question of editing on multicolorings of complete bipartite graphs.

Alon and Stav cite [20] in their own papers on edit distance, which they pursued independently. They generalize the result on $d^*(H)$ to define $d^*(\mathcal{H})$ for a general hereditary property and show that the computation of d^* can be accomplished by restricting attention to the Erdős-Rényi random graph $G_{n,p}$.

In [19], written with József Balogh, we define a function of a hereditary property \mathcal{H} and of $p \in [0, 1]$ that computes, asymptotically, the distance of $G_{n,p}$ from \mathcal{H} . We also determine properties of the edit distance function that can help to determine $d^*(\mathcal{H})$. We compute the edit distance function for several hereditary properties, showing that for any rational number, $r \in [0, 1]$, there is a hereditary property \mathcal{H} such that the maximum of the function $(p^*(\mathcal{H}), d^*(\mathcal{H}))$ has $p^*(\mathcal{H}) = r$. Irrational numbers are possible also: $p^*(\text{Forb}(K_{3,3})) = \sqrt{2} - 1$. In the process, we developed a so-called weighted Turán lemma that may have some independent applications.

In Progress. I intend to continue research in this area. In upcoming work with Balogh, we intend to compute the edit distance function for a large class of hereditary properties, using some of the optimal weighting techniques in the paper mentioned in the above paragraph. Moreover, the computation of an edit distance function is difficult for an arbitrary hereditary property and doing so should lead to a better general understanding of hereditary properties. In the long run, I hope to generalize the theory of functions of hereditary properties to compare the edit distance function to the following function introduced by Bollobás and Thomason:

$$c_p(\mathcal{H}) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} -\log_2 \Pr(G_{n,p} \in \mathcal{H}) / \binom{n}{2}.$$

Graph tiling.

The landmark papers of Hajnal and Szemerédi, Erdős and Stone, Erdős and Simonovits, Alon and Yuster as well as Komlós, Sárközy and Szemerédi are the basis of the theory of tiling graphs. An H -tiling of a graph G is a spanning subgraph that consists of vertex-disjoint copies of H .

The theorem of Hajnal and Szemerédi gives that if a graph G has n vertices, minimum degree at least $(k-1)n/k$ and $k|n$, then G has a K_k -tiling. The case $k=3$ was first proved by Corrádi and Hajnal.

In the multipartite version of Hajnal-Szemerédi, G is an r -partite graph with N vertices in each part and the problem is to find a minimum-degree condition that would ensure a K_r -tiling. Fischer conjectured in 1999 a similar degree condition to that of Hajnal-Szemerédi. That is, if each vertex of G is adjacent to at least $(r-1)N/r$ vertices in each of the $r-1$ other parts of the r -partition, then G has a K_r -factor. For ease of notation, if such a graph has each vertex adjacent to at least $(r-1)N/r$ vertices in each part, then we write that $\bar{\delta}(G) \geq (r-1)N/r$.

If true, this implies Hajnal-Szemerédi asymptotically. That is, for any $\epsilon > 0$, if a graph G has minimum-degree at least $(r-1)N/r + \epsilon N$, then a random r -partitioning of the vertex set will yield a subgraph G' which is r -partite, has N vertices in each part, and $\bar{\delta}(G') \geq$

$(r - 1)N/r$. A K_r -tiling of G' yields a K_r -tiling of G .

In [3], written with Csaba Magyar, we show that there exists an integer N_0 such that if a tripartite graph G with $N \geq N_0$ vertices in each part has the property that every vertex has at least $2N/3$ neighbors in each of the 2 other parts, then G has a K_3 -tiling unless G is a unique exceptional graph.

This exceptional graph can be generalized to show that Fischer's conjecture is not true if both r and N are odd. However, it is the only exception known and $\bar{\delta}(G) = (r - 1)N/r$ in this case. Hence, the conjecture may be correct if the sufficient condition is, instead $\bar{\delta}(G) \geq (r - 1)N/r + 1$.

In [17], written with Endre Szemerédi, we show that there exists an integer N_0 such that if a quadripartite graph G with $N \geq N_0$ vertices in each part has the property that every vertex has at least $3N/4$ neighbors in each of the 3 other parts, then G has a K_4 -tiling. There is no exceptional graph.

Komlós, Sárközy and Szemerédi improve a result of Alon and Yuster to give necessary minimum degree conditions to ensure an H -tiling in a large graph. That is, for every graph H with chromatic number χ , there exist integers N_0 and C such that if G has $N \geq N_0$ vertices, N a multiple of $|V(H)|$, and minimum degree at least $(1 - 1/\chi)N + C$, then G has an H -tiling.

In [23], written with Yi Zhao, we show that for every h , there exists an N_0 such that if G is a tripartite graph with $N \geq N_0$ vertices, N a multiple of $6h$ and $\bar{\delta}(G) \geq 2N/3 + h - 1$, then G has a $K_{h,h,h}$ -tiling. Moreover, this is best possible. We also prove that $\bar{\delta}(G) \geq 2N/3 + 2h - 1$ is sufficient, provided N is a multiple of h .

The techniques in the above-mentioned papers involve using Szemerédi's regularity lemma and the so-called Blow-up lemma as well as a number of auxiliary lemmas which were developed explicitly to obtain these results.

In a similar vein in [10], written with Axenovich, we prove a result on the so-called strong chromatic number of a graph. A graph, G , is *strongly r -colorable* if, after adding $r\lceil n/r \rceil - n$ isolated vertices to G and considering any partition of the resulting graph into disjoint subsets of size r , then there is a proper r -coloring of the graph such that each part contains exactly one vertex of each color. The *strong chromatic number* of G , $\chi_S(G)$, is the least such r . Haxell improved a result of Alon, proving that if G has maximum degree Δ , then $\Delta + 1 \leq \chi_S(G) \leq 3\Delta - 1$. We use techniques reminiscent of the Almost-covering lemma of [3,23] (see also work by Fischer and by R. Johansson) to prove that if $\Delta(G) \geq n/6$, then $\chi_S(G) \leq 2\Delta$ and this bound is tight.

Smoothed analysis of graphs.

In [4], written with Tom Bohman and Alan Frieze and in [8], written with Bohman, Frieze and Michael Krivelevich, we introduced a random graph model in which a graph G is chosen arbitrarily from a family of graphs \mathcal{G} and then a small set R of random edges is added to G . The resulting graph is analyzed for the properties of the resulting graph. In both papers, \mathcal{G} is the family of graphs with n vertices and minimum degree at least dn .

The main result of [4] is that if $|R| \geq (30 \ln d^{-1} + 13)n$, then the resulting graph is Hamiltonian with high probability, and for $d \leq 1/10$ there is a $G \in \mathcal{G}$ such that adding $(\ln d^{-1}/3)n$ random edges results in a non-Hamiltonian graph, with high probability. In [8],

there are a number of results, one of the more striking is that if $G \in \mathcal{G}$ and $|R| \rightarrow \infty$ as $n \rightarrow \infty$, then the resulting graph has diameter at most 5, with high probability.

Szemerédi's regularity lemma features prominently in most of the theorem proofs. There have been two subsequent papers by other authors on this model of which I am aware, one by Krivelevich, Sudakov and Tetali, another by Sudakov and Vondrák.

Online intersecting hypergraphs.

In [6], written with Bohman, Colin Cooper, Frieze and Miklós Ruszinkó, the online intersecting hypergraph process is introduced. Reminiscent to the more well-studied triangle-free graph process, in the online hypergraph process, there is an underlying vertex set, $\{1, \dots, n\}$ and subsets of size r (hyperedges) are chosen as follows: e_1 is arbitrary, e_2 is chosen uniformly at random from all hyperedges that have nonempty intersection with e_1 . In general, e_{i+1} is chosen uniformly at random from all hyperedges that have nonempty intersection with each of e_1, \dots, e_i . The process continues until the set of eligible hyperedges is exhausted.

Let $\mathcal{E}_{n,r}$ be the event that the resulting hypergraph is an Erdős-Ko-Rado family; i.e, a family that fixes a single vertex and is of size $\binom{n-1}{r-1}$. The result of [6] is a very sharp threshold as to how large r must be in order for the resulting hypergraph to fail to be maximal. If $r = c_n n^{1/3} < n/2$, then

$$\lim_{n \rightarrow \infty} \Pr(\mathcal{E}_{n,r}) = \begin{cases} 1, & \text{if } c_n \rightarrow 0; \\ 1/(1 + c^3), & \text{if } c_n \rightarrow c; \text{ and} \\ 0, & \text{if } c_n \rightarrow \infty. \end{cases}$$

In [14], written with Bohman, Cooper, Ruszinkó and Cliff Smyth, we extend this result to consider the process when r is in the range $\omega(n^{1/3}) = r = o(n^{5/12})$. The description of the hypergraph is complex, but the theorem states that the structure of the resulting hypergraph is determined, with high probability, by the outcome of a single random variable.

The techniques involved in both papers involve exploiting the early behavior of the process, giving a very high-degree vertex in the first few steps. The hypergraph process builds on this vertex to the point at which the process has only one possible conclusion, with high probability.

Positional games.

Recently, I have begun some research into the theory of positional games. In [21], written with Balogh and András Pluhár, we study the diameter game. That is, this is a so-called Maker-Breaker game in which each player selects edges in turn, Maker seeking to create a graph, say, of diameter at most 2 and Breaker wants to prevent this. It is easy to see that if each player gets one move, Breaker can win this game. If, however, Maker gets 2 moves per round, but Breaker gets $Cn^{1/8}/(\ln n)^{1/2}$ per round, for some constant C , then Maker has a winning strategy for n sufficiently large.

This has implications with respect to the so-called probabilistic intuition, in which Maker is expected win the game to achieve a graph with diameter at most 2 if his final proportion of

edges is enough so that an Erdős-Rényi random graph with Maker's density has diameter 2 with high probability. This intuition fails if Maker gets one move, but if the game is accelerated, then the intuition is at least partially restored. We also investigate the diameter game for diameters greater than 2, observing a similar acceleration phenomenon. The techniques involve a careful analysis of an explicit strategy of Maker, alternately attempting to achieve different properties of his graph.

In [23], written with Balogh, we investigate an Avoider-Enforcer game, in which Avoider tries to avoid his graph having monotone property \mathcal{P} as long as possible and Enforcer makes moves in order to force Avoider's graph to have property \mathcal{P} as soon as possible. The smallest integer t such that Enforcer can win this game is $\tau_E(\mathcal{P})$. The technique involves exploiting the game of JumbleG by Frieze, Krivelevich, Pikhurko and Szabó and is, as far as we know, the first paper on positional games to use Szemerédi's regularity lemma.

Weighted Ramsey.

My collaborations with Maria Axenovich have been productive and quite diverse. I wish to focus on one additional paper in this section. In [16], we prove a number of results regarding weighted Ramsey numbers, first defined by Fujisawa and Ota. The formulation we use is that the number $wR(n, k)$ is the minimum q such that there is an assignment of nonnegative real numbers to the edges of K_n so that their sum is $\binom{n}{2}$ and there is a Red/Blue coloring of these edges such that for any set, H of k vertices, the sum of the weights on the Red edges with both endpoints in H is at most q and the sum of the weights on the Blue edges with both endpoints in H is at most q .

The problem relates closely to a classical problem of packing triangles in a graph and its complement, first asked by Erdős. In fact, weighted Ramsey is a generalization of the problem of packing cliques in a graph and its complement. We directly use some results on the triangle-packing problem by Keevash and Sudakov and a more general result on fractional graph packing by Haxell and Rödl to improve the results of Fujisawa and Ota. We show that the weighted Ramsey question is equivalent to the solution of a linear program. The most general result of the paper combines linear programming duality, Szemerédi's regularity lemma and current bounds on the so-called diagonal Ramsey numbers.

This document and my CV can be found at my main web page:

<http://orion.math.iastate.edu/rymartin>

A shorter document which lists my publications can be found at:

<http://www.math.iastate.edu/rymartin/cv/pubs.pdf>

Both documents are hyperlinked, providing links to the journal versions of my papers or to preprints.