Summary of smoothed analysis results

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For a positive integer n and $d \in [0,1]$, let $\mathcal{G}(n,d)$ be the set of graphs on n vertices with minimum degree at least d. Let G be constructed by taking $H \in \mathcal{G}(n,d)$ and adding a set R of random edges, $R \subseteq [n]^2 \setminus E(H)$, $m = |R|$. We write $G = H + R$.

The term **whp** means that a property occurs **with high probability**. That is, the property occurs with probability approaching 1 as $n \to \infty$.

1 Hamiltonicity

A graph is said to be **Hamiltonian** if it contains a spanning cycle.

Theorem 1.1 (Bohman-Frieze-M. [1], 2003) *Let $0 < d \leq 1/2$ and let $\theta = \ln d^{-1}$. Note that $\theta \geq 0.69$.*

(a). *If $m \geq (30\theta + 13)n$, then G is Hamiltonian **whp**.*

(b). *For $d \leq 1/10$, there exist graphs $H_0 \in \mathcal{G}(n,d)$ such that if $m < \theta n/3$, then **whp** G is not Hamiltonian.*

Theorem 1.2 (Bohman-Frieze-M. [1], 2003) *Suppose $H \in \mathcal{G}(n,d)$ and $1 \leq \alpha < d^2 n/2$ and so $d > n^{-1/2}$ (d need not be constant in this theorem). If*

$$\frac{md^3}{\ln d^{-1}} \to \infty,$$

*then G is Hamiltonian **whp**.*

Theorem 1.3 (Bohman-Frieze-M. [1], 2003) *Suppose $0 < d < 1/2$ is constant. Let $\mathcal{D}(n,d)$ denote the family of directed graphs H on n vertices such that $\delta^+(H), \delta^-(H) \geq dn$. Let $D = H + R$, where $|R| = m$ is chosen randomly from $[n]^2 \setminus E(H)$. Again, $\theta = \ln d^{-1}$.*

(a). *If* $m \geq \left(d^{-1}(15+6\theta)+5d^{-2}\right) n$*, then* D *is Hamiltonian* ***whp****.*

(b). *For* $d \leq 1/10$*, there exist directed graphs* $H_0 \in \mathcal{D}(n,d)$ *such that if* $m < \theta n/3$ *then* ***whp*** D *is not Hamiltonian.*

2 Small cliques

The graph K_r is the complete graph on r vertices.

Theorem 2.1 (Bohman-Frieze-Krivelevich-M. [2], 2004) *Let* $r > r_0 \geq 2$ *be integers and* $d \in \left(\frac{r_0-2}{r_0-1}, \frac{r_0-1}{r_0}\right]$ *be a fixed constant.*

(a). *If* $m = \omega\left(n^{2-2/(\lceil r/r_0\rceil-1)}\right)$*, then* G *contains a copy of* K_r ***whp****.*

(b). *There exists a graph* $H_0 \in \mathcal{G}(n,d)$ *such that if* $m = o\left(n^{2-2/(\lceil r/r_0\rceil-1)}\right)$*, then* ***whp*** G *fails to contain* K_r*.*

3 Diameter

The distance between two vertices is the length of the shortest path between them. The diameter of a graph G, denoted $\operatorname{diam}(G)$, is the maximum distance between any pair of distinct vertices.

It is easy to see the following:

Proposition 1 *If* $1/2 \leq d < 1$*, then* $\operatorname{diam}(H) = 2$*.*

Theorem 3.1 (Bohman-Frieze-Krivelevich-M. [2], 2004) *Let* $d \in (0, 1/2)$ *be a fixed constant. If* $m = \omega(1)$*, then* $\operatorname{diam}(G) \leq 5$ ***whp****.*

Theorem 3.2 (Bohman-Frieze-Krivelevich-M. [2], 2004) *Let* $d \in (0, 1/2)$ *be a fixed constant.*

(a). *If* $m = \frac{1-d}{d^2}\ln n + \omega(1)$*, then* $\operatorname{diam}(G) \leq 3$ ***whp****.*

(b). *There exists a graph* $H_0 \in \mathcal{G}(n,d)$ *such that if* $m = \frac{\log n}{-2\log(1-2d)} - \omega(1)$*, then* ***whp*** $\operatorname{diam}(G) \geq 5$*.*

Theorem 3.3 (Bohman-Frieze-Krivelevich-M. [2], 2004) *Let* $d \in (0, 1/2)$ *be a fixed constant.*

(a). *If* $m = \frac{1-d}{d} n \ln n + \omega(1)$*, then* $\operatorname{diam}(G) \leq 2$ ***whp****.*

(b). *There exists a graph* $H_0 \in \mathcal{G}(n,d)$ *such that if* $m = n\log n/2 - \omega(1)$*, then* ***whp*** $\operatorname{diam}(G) \geq 3$*.*

4 Connectivity

Theorem 4.1 (Bohman-Frieze-Krivelevich-M. [2], 2004) *Let $d \in (0, 1/2)$.*

(a). *If $k = O(1)$ and $m = \omega(1)$, then G is k-connected **whp**. If $\omega(1) \leq k \leq d^2 n/32$ and $m = 640k/d^2$, then G is k-connected **whp**.*

(b). *There exists a graph $H_0 \in \mathcal{G}(n, d)$ such that if $m \leq \frac{k}{2} \left\lfloor \frac{n}{dn+1} \right\rfloor$, then **whp**, G fails to be k-connected.*

5 Existence of subgraphs

For a graph Γ, denote $v(\Gamma)$ to be the number of vertices and $e(\Gamma)$ to be the number of edges. Then $m(\Gamma)$ is defined to be

$$m(\Gamma) = \max \left\{ \frac{e(\Gamma)}{v(\Gamma)} : \Gamma' \subseteq \Gamma, v(\Gamma') > 0 \right\}.$$

This is half of MAD, the Maximum Average Degree. For every positive integer r, define

$$m_r(\Gamma) = \min_{V(\Gamma) = \cup_i V_i} \max_i m\left(\Gamma[V_i]\right).$$

Here, we instead consider $H \in \mathcal{G}_a(n, d)$, where $\mathcal{G}_a(n, d)$ is the family of graphs on n vertices with average degree at least d. I.e., $e(H) \geq dn/2$.

Theorem 5.1 (Krivelevich-Sudakov-Tetali [3], 2006?) *Let $d \in (0, 1)$ be a fixed constant and let $r \geq 2$ be the unique integer satisfying $d \in \left(\frac{r-2}{r-1}, \frac{r-1}{r} \right]$. Let Γ be a fixed graph.*

(a). *If $m = \omega\left(n^{2-1/m_r(\Gamma)}\right)$, then G contains a copy of Γ **whp**.*

(b). *There exists a graph $H_0 \in \mathcal{G}_a(n, d)$ with $d = (1 + o(1))\frac{r-1}{r}$ such that if $m = o\left(n^{2-1/m_r(\Gamma)}\right)$, then **whp**, G fails to contain a copy of Γ.*

6 Ramsey questions

Theorem 6.1 (Krivelevich-Sudakov-Tetali [3], 2006?) *Let $d \in (0, 1)$ be a fixed constant.*

(a). *If $m = \omega\left(n^{2-2/(t-1)}\right)$, then $G \to (K_3, K_t)$ **whp**.*

(b). *For every $t \geq 3$, there exists a graph H_0 on n vertices with $n^2/4$ edges such that if $m = o\left(n^{2-2/(t-1)}\right)$, then **whp**, G has a Red-Blue edge coloring with no Red copy of K_3 and no Blue copy of K_t.*

7 k-SAT

Given n boolean variables $x_1, \ldots, x_n$, a literal is a variable x_i or its negation $\overline{x_i}$ for $i = 1, \ldots, n$. For a set of literals Y, denote by $\overline{Y}$ the set of literals $\{\overline{y} : y \in Y\}$. A disjunction of k literals is called a k-clause. Let $\mathcal{C}_k$ be the set of all k-clauses over $x_1, \ldots, x_n$; i.e., the set of $2^k\binom{n}{k}$ possible disjunctions of k distinct, non-complementary literals. A k-SAT formula F is a conjunction of clauses from $\mathcal{C}_k$.

Theorem 7.1 (Krivelevich-Sudakov-Tetali [3], 2006?) **(a)**. *Let $c > 0$, $0 \le \epsilon < 1/k$, and let F be a satisfiable k-SAT formula on n variables with at least $cn^{k-\epsilon}$ clauses. Then almost surely a conjunction of F with $m = \omega\left(n^{k\epsilon}\right)$ random clauses from $\mathcal{C}_k$ is not satisfiable.*

(b). *For every $0 \le \epsilon \le 1/k$, there exists a k-SAT formula on n variables with $\Omega\left(n^{k-\epsilon}\right)$ clauses such that a conjunction of F with $m = o\left(n^{k\epsilon}\right)$ random clauses from $\mathcal{C}_k$ is **whp** satisfiable.*

8 Hypergraphs

Theorem 8.1 (Sudakov-Vondrak [4], 2006?) **(a)**. *Let $k, \ell \ge 2$, $\epsilon \ge 0$ be fixed and let H be a 2-colorable k-uniform hypergraph with $\Omega\left(n^{k-\epsilon}\right)$ edges. Then the hypergraph H' obtained by adding to H a collection R of $\omega\left(n^{\ell\epsilon/2}\right)$ random ℓ-tuples is non-2-colorable **whp**.*

(b). *For fixed $k, \ell \ge 2$ and $0 \le \epsilon < 2/\ell$, there exists a 2-colorable k-uniform hypergraph H with $\Omega\left(n^{k-\epsilon}\right)$ edges such that its union with a collection R of $o\left(n^{\ell\epsilon/2}\right)$ random ℓ-tuples is **whp** 2-colorable.*

References

[1] Bohman, T., Frieze, A., and Martin, R. How many random edges make a dense graph hamiltonian?, *Random Structures Algorithms*, **22**, no. 1, 33–42.

[2] Bohman, T., Frieze, A., Krivelevich, M. and Martin, R. Adding random edges to dense graphs, *Random Structures Algorithms*, **24**, no. 2, 105–117.

[3] Krivelevich, M., Sudakov, B. and Tetali, P. On smoothed analysis in dense graphs and formulas, *Random Structures Algorithms*, to appear.

[4] Sudakov, B. and Vondrák, J. How many random edges make a dense hypergraph non-2-colorable? preprint