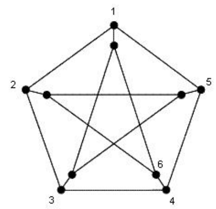
Let G be a graph. A subset D of the vertex set $V(G)$ is a *dominating set* if every vertex $v \in V \setminus D$ is adjacent to a vertex in D.

EXAMPLE Petersen graph P

- If $D = V(P)$, then D is dominating set (trivially).
- If $D = \{1, 2, 3, 4, 5\}$ (the outer cycle), then D is a dominating set.
- If $D = \{3, 5, 6\}$, then D is a dominating set.



EXAMPLE $K_{m,n}$

The minimal dominating set of $K_{m,n}$ has 2 vertices, one from each partition.

Let $\gamma(G)$ be the domination number, the minimal number of vertices required for a dominating set of G.

RESULTS ON DOMINATION NUMBER

Ore: if $\delta(G) \geq 1$, then $\gamma(G) \leq n/2$

Blank: if $\delta(G) \geq 2$, then $\gamma(G) \leq 2n/5$

Reed: if $\delta(G) \geq 3$, then $\gamma(G) \leq 3n/8$

From Reed's result, since $\delta(P) = 3$, we have $\gamma(P) \leq 3(10)/8 < 4$.

THEOREM Let $G = (V, E)$ be a graph on n vertices, with minimum degree $\delta > 1$. Then, G has a dominating set of at most $n\dfrac{1 + \ln(\delta + 1)}{\delta + 1}$ vertices.

PROOF: Let $p \in [0, 1]$, to be chosen later. Choose at random and independently each vertex from V with probability p. If chosen, place vertex v in set X. Let Y be the set of vertices that are not in X and are not neighbors to any vertex in X.

Then, E$[|X|] = np$. Also, for fixed $v \in V$,

$$\begin{aligned} \Pr[v \in Y] &= \Pr[v \notin X \text{ and the neighbors of } v \notin X] \\ &\leq (1-p)^{\delta+1} \end{aligned}$$

Since Y is a random variable, so is $|Y|$. Let $\mathbf{1}_v$ be the indicator variable with

$$\mathbf{1}_v = \begin{cases} 1 & \text{if } v \in Y \\ 0 & \text{if } v \notin Y \end{cases}$$

Then, $|Y| = \sum_{j=1}^{n} \mathbf{1}_j$ and $\mathrm{E}[|Y|] = \mathrm{E}\left[\sum_{j=1}^{n} \mathbf{1}_j\right] = \sum_{j=1}^{n} \mathrm{E}[\mathbf{1}_j] = \sum_{j=1}^{n} \Pr[j \in Y] \leq \sum_{j=1}^{n} (1-p)^{\delta+1} = n(1-p)^{\delta+1}$
Hence, $\mathrm{E}[|X| + |Y|] = \mathrm{E}[|X|] + \mathrm{E}[|Y|] \leq np + n(1-p)^{\delta+1}$.

Set set $D = X \cup Y$. Thus, D is a dominating set of G.

Finally, we optimize the value of p. Using the bound $1 - p \leq e^{-p}$ (Taylor Series approximation from calculus), we have

$$|D| \leq np + n(1-p)^{\delta+1} \leq np + ne^{-p(\delta+1)}.$$

Let $f(p) = np + ne^{-p(\delta+1)}$. Thus, $f'(p) = n - n(\delta+1)e^{-p(\delta+1)}$. Setting this equal to zero, we have

$$\begin{aligned}
0 &= n - n(\delta+1)e^{-p(\delta+1)} \\
1 &= (\delta+1)e^{-p(\delta+1)} \\
\frac{1}{\delta+1} &= e^{-p(\delta+1)} \\
-p(\delta+1) &= \ln\left(\frac{1}{\delta+1}\right) \\
p &= -\frac{\ln(1/(\delta+1))}{\delta+1} \\
p &= \frac{\ln(\delta+1)}{\delta+1}
\end{aligned}$$

With $p = \dfrac{\ln(\delta+1)}{\delta+1}$, we have

$$\begin{aligned}
|D| &\leq n\left(\frac{\ln(\delta+1)}{\delta+1}\right) + ne^{-\left(\frac{\ln(\delta+1)}{\delta+1}\right)(\delta+1)} \\
&= n\left(\frac{\ln(\delta+1)}{\delta+1} + \frac{1}{\delta+1}\right) \\
&= n\frac{1 + \ln(\delta+1)}{\delta+1}
\end{aligned}$$