

1 Subadditivity and Fekete's theorem

Lemma 1 (Fekete) *If $\{a_n\}$ is subadditive then $\lim_{n \rightarrow \infty} \frac{a_n}{n}$ exists and equals the $\inf_{n \rightarrow \infty} \frac{a_n}{n}$.*

Recall that $\{a_n\}$ is subadditive if $a_{m+n} \leq a_m + a_n$.

The goal would be to show that $\{\log R(k, k)\}_{k=3}^{\infty}$ is subadditive.

$$k \log \sqrt{2} \lesssim \log R(k, k) \lesssim k \log 4$$

Example: Shannon capacity is subadditive.

2 The Chung-Lu model

Now, to continue the Chung-Lu model from last time,

Theorem 2.1 (Chung-Lu [1]) *For a random graph $G(\vec{w})$ where \vec{w} is admissible, the average distance is $(1 + o(1)) \frac{\log n}{\log \tilde{d}}$ with high probability, where \tilde{d} is the second order average degree, denoted $\frac{\sum w_i^2}{\sum w_i}$.*

Corollary 1 (Chung-Lu [1]) *If $np \geq C > 1$ and $\frac{\log n}{\log np} \rightarrow \infty$ as $n \rightarrow \infty$, then the average distance of $G(n, p)$ approaches $\frac{\log n}{\log np}$ as $n \rightarrow \infty$.*

In the power-law graph with parameter β , the number of vertices of degree k is proportional to $\frac{1}{k^\beta}$.

Theorem 2.2 (Chung-Lu [1]) *If $\beta > 3$, average distance is $(1 + o(1)) \frac{\log n}{\log \tilde{d}}$ and diameter $= \theta(\log(n))$ with high probability.*

This is not tremendously useful for internet model power-law graphs since most internet graphs are $2 < \beta < 3$, but this result is very nice and clean.

Theorem 2.3 (Chung-Lu [1]) *Let $d > 1$. Let m be maximum degree. With high probability, $\maxdegree < \max_i \{w_i + c\sqrt{w_i}\}$ for any $c \rightarrow \infty$ and $\log m \gg \frac{\log n}{\log \log n}$.*

If $2 < \beta < 3$, average distance is $\leq (2 + o(1)) \frac{(\log \log n)}{(-\log(\beta - 2))}$. Note that $\beta = 2$ is a “huge logjam.” Also, diameter = $\theta(\log n)$ with high probability.

If $\beta = 3$, average distance = $\theta\left(\frac{\log n}{\log \log n}\right)$ and diameter = $\theta(\log n)$, both with high probability, and note that all these logs have base e .

3 Applications of Chung-Lu

Consider a graph whose vertices represent published mathematicians and whose edges represent two authors publishing together. Call these Authoring Graphs. Denote G_1 as the graph which has two authors connected if they coauthor. G_2 has two authors connected only if they were the only two authors.

Based on the data from the Erdős Number Project [2],

- $n \approx 337,000$: That is, there were about 337,000 authors considered.
- $m_1 \approx 496,000$: That is, the number of edges in G_1 is 496,000.
- $m_2 \approx 226,000$: That is, the number of edges in G_2 is 226,000.
- Note that G_2 is thus disconnected, as there are fewer edges than vertices.
- $\beta_1 = 2.97$, $d_1 = 2.94$
- $\beta_2 = 3.26$, $d_2 = 1.34$

Theorem 2.3 predicts that the average distance ≈ 9.89 . The actual measured value is 9.56.

Next is the preferential attachment model (also called the scale-free model) of the Internet. It doesn't exactly work right, but we work with it anyway. Construct this by attaching v_{i+1} to v_j with probability proportional to $\frac{1}{\deg(v_j)^\beta}$.

Papadimitriou has presented various papers in this area. Statistics of Internet graphs have been presented by M., P. and C. Faloutsos [3] in 1999.

Theorem 3.1 Given $G(\vec{w})$, m is the maximum degree, $\tilde{d} > \sqrt{m} \log n$. Then $\lambda_{\max}(G)$ is $(1 + o(1))\tilde{d}$ with high probability. We may assume $m \leq \max_i \{w_i + c\sqrt{w_i}\}$ as $c \rightarrow \infty$.

Recall from the April 4-6, 2006 notes on expander graphs, $\frac{2e}{n} \leq \lambda_{\max} \leq \delta$.

Fact 1 Let A be a symmetric, nonnegative $n \times n$ matrix. For any positive constants c_1, \dots, c_n ,

$$\lambda_{\max}(A) \leq \max_{1 \leq i \leq n} \left\{ \frac{1}{c_i} \sum_{j=1}^n c_j a_{ij} \right\}.$$

Proof. Let

$$C = \begin{bmatrix} c_1 & 0 & 0 & \dots \\ 0 & c_2 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_n \end{bmatrix}$$

and look at $C^{-1}AC$ juxtaposed with A .

Claim 1: $C^{-1}AC$ and A have the same spectrum.

A is symmetric, so $A = P^{-1}DP$ for some diagonal D and invertible P . So,

$$C^{-1}AC = C^{-1}(P^{-1}DP)C = (PC)^{-1}D(PC)$$

and thus $C^{-1}AC$ is diagonalizable with the same diagonal entries.

So, find $\lambda_{\max}(A) = \lambda_{\max}C^{-1}AC$. Note that $(AC)_{kl} = a_{kl}c_l$ so

$$(C^{-1}AC)_{ij} = \frac{1}{c_i}(AC)_{ij} = \frac{1}{c_i}a_{ij}c_j$$

So, $\frac{1}{c_i} \sum_{j=1}^n c_j a_{ij}$ is the i^{th} row sum of $C^{-1}AC$. Let \vec{x}_{\max} be an eigenvector corresponding to λ_{\max} . So,

$$(C^{-1}AC)\vec{x}_{\max} = \lambda_{\max}\vec{x}_{\max}.$$

Then $\frac{1}{c_i} \sum_{j=1}^n c_j a_{ij} x_j = \lambda_{\max} x_i \forall i$. Let $x_1 \geq x_i \forall i$ **wlog**. So,

$$\lambda_{\max} x_i = \frac{1}{c_i} \sum_{j=1}^n c_j a_{ij} x_j \leq \frac{x_1}{c_i} \sum_{j=1}^n c_j a_{ij} = \lambda_{\max} x_1$$

and

$$\lambda_{\max} \leq \frac{x_1}{x_i} \frac{1}{c_i} \sum_{j=1}^n c_j a_{ij}.$$

Thus for $i = 1$,

$$\lambda_{\max} \leq \frac{1}{c_i} \sum_{j=1}^n c_j a_{1j} \leq \max_i \left\{ \sum_{j=1}^n c_j a_{ij} \right\}$$

□

Lemma 2 $|X_i| \leq M$ with probability 1, but be aware this is on a set of measure zero.

$$\text{Let } X = \sum X_i. \text{ Pr}(|X - E(x)| > a) \leq \exp \left\{ \frac{a}{2\text{Var}(X)} + \frac{Ma}{3} \right\}$$

Lemma 3 Let $G(\vec{w})$ denote the random graph with expected degree sequence \vec{w} .

$$\lambda_{\max}(G(\vec{w})) \leq \tilde{d} + \sqrt{6\sqrt{m \log n}(\tilde{d} + \log n)} + 3\sqrt{m \log n}$$

with high probability.

$$\text{In particular, } \lambda_{\max} < 2\tilde{d} + 6\sqrt{m \log n}.$$

References

- [1] F.R.K. Chung, L. Lu, The average distances in random graphs with given expected degrees. *Internet Math.* **1** (2003), no. 1, 91–113. (<http://www.math.ucsd.edu/~fan/wp/avef.pdf>)
- [2] J. Grossman, The Erdős Number Project, <http://www.oakland.edu/~grossman/erdoshp.html>
- [3] M. Faloutsos, P. Faloutsos, and C. Faloutsos, On power-law relationships of the Internet topology, *Proceedings SIGCOMM*, 1999, 251-262.