THEOREM : $\Pr(X > E[X] + \lambda) \le \exp\left(-\frac{\lambda^2}{2(\nu + a\lambda/3)}\right)$ with $X = \sum a_i X_i$, $p_i = \Pr(X = 1), 1 - p_1 = \Pr(X = 0), a = \max_i a_i$, and $\nu = \sum_{i=1}^{n} a_i^2 p_i$.

Recall that $E[\deg(v_i)] = w_i$ with the w_is chosen beforehand.

Let $S \subseteq V(G)$ and $\text{Vol}(S) = \sum_{i \in X} w_i$.

FACT 2: With probability at least $1 - 2e^{t^2/2}$ we have $2e(G) > \text{Vol}(G) - c\sqrt{\text{Vol}(G)}$. (Note that $\text{Vol}(G)$ is just $\text{Vol}(S)$ where $S = V(G)$).

So from the above theorem,

$$\Pr(2e(a) < (1+\epsilon)\text{Vol}(G)) > 1 - \exp\left(\frac{-\epsilon^2 \text{Vol}(G)}{2 + 2\epsilon/3}\right)$$

Interesting graphs start to occur around $\text{Vol}(G) = \Omega(n)$.

The volume is bounded by $2 \le \text{Vol}(G) \le 2\binom{n}{2}$.

NOTE: Recall $d = \frac{\sum w_i}{n} = \frac{\text{Vol}(G)}{n}$ which is the average degree. Also recall that

$$\begin{aligned}
\tilde{d} &= \frac{\sum w_i^2}{\sum w_i} \\
&= \frac{\sum w_i^2}{\text{Vol}(G)} \\
&= \frac{\frac{1}{n}\sum w_i^2}{\frac{1}{n}\sum w_i} \\
&\ge \frac{(\frac{\sum w_i}{n})^2}{\frac{1}{n}\sum w_i} \qquad (1) \\
&= \frac{\sum w_i}{n} \\
&= d
\end{aligned}$$

With (1) coming from Jansen's inequality. If all the w_is are equal, this is an equal sign.

EXAMPLE 3: Let $d < 1 < \tilde{d}$ on a graph with n vertices, M large. For $\left\lceil n - \frac{n}{M} \right\rceil$ of the vertices, the weight is $x = o(1)$ as $n \to \infty$ and M remaining constant.

The other vertices have weight $1+\epsilon$. Then

$$\begin{aligned}
d &= \frac{x\left\lceil n-\frac{n}{M}\right\rceil+(1+\epsilon)\left\lfloor\frac{n}{M}\right\rfloor}{n}\\
&\approx \frac{xn+(1+\epsilon-x)\frac{n}{m}}{n}\\
&\approx \frac{x+1+\epsilon-x}{M}\\
&\to \frac{1+\epsilon}{M} \quad \text{As } n\to\infty
\end{aligned}$$

Also

$$\begin{aligned}
\tilde{d} &= \frac{x^2\left\lceil n-\frac{n}{M}\right\rceil+(1+\epsilon)^2\left\lfloor\frac{n}{M}\right\rfloor}{x\left\lceil n-\frac{n}{M}\right\rceil+(1+\epsilon)\left\lfloor\frac{n}{M}\right\rfloor}\\
&\approx \frac{x^2(n-\frac{n}{M})+(1+\epsilon)^2\frac{n}{M}}{x(n-\frac{n}{M})+(1+\epsilon)\frac{n}{M}}\\
&\approx \frac{x^2(1-\frac{1}{M})+(1+\epsilon)^2\frac{1}{M}}{x(1-\frac{1}{M})+(1+\epsilon)\frac{1}{M}}\\
&\to 1+\epsilon \quad \text{As } n\to\infty
\end{aligned}$$

Then $\text{Vol}(G)=nd\approx n\left(\frac{1+\epsilon}{M}\right)$.

Let A be the set of vertices with weight x. Then $|A|=\left\lceil n-\frac{n}{M}\right\rceil$. Let B be the set of vertices with weight $1+\epsilon$. Then $|B|=\left\lfloor\frac{n}{m}\right\rfloor$.

Let $u_iv_j\in A$. Then

$$\begin{aligned}
\Pr(v_i\sim v_j) &= \frac{w_iw_j}{\sum v_k}\\
&= \frac{x^2}{\text{Vol}(G)}\\
&\approx \frac{x^2}{n(\frac{1+\epsilon}{M})}
\end{aligned}$$

Then A has an Erdős-Rényi graph with $\left\lceil n-\frac{n}{M}\right\rceil$ vertices.

Then the number of vertices $\left\lceil n-\frac{n}{M}\right\rceil\approx n\left(\frac{M-1}{M}\right)$ and

$$\begin{aligned} p &= \frac{x^2}{\mathrm{Vol}(G)} \\ &\approx \frac{x^2 M}{n(1+\epsilon)} \end{aligned}$$

So we have

$$\begin{aligned} np &= n\frac{M-1}{M} \cdot \frac{x^2 M}{n(1+\epsilon)} \\ &= \frac{(M-1)x^2}{1+\epsilon} \end{aligned}$$

Also B has an Erdős-Rényi graph with $\left\lfloor \frac{n}{M} \right\rfloor$ vertices. In B we have

$$\begin{aligned} p &= \frac{(1+\epsilon)^2}{\mathrm{Vol}(G)} \\ &\approx \frac{(1+\epsilon)^2}{n\frac{1+\epsilon}{M}} \\ &\approx \frac{M(1+\epsilon)}{n} \end{aligned}$$

So we have

$$\begin{aligned} np &\approx \frac{n}{M} \cdot \frac{M(1+\epsilon)}{n} \\ &= 1+\epsilon \\ &> 1 \end{aligned}$$

So we know that $\exists C = C(\epsilon)$ where B contains a component of size greater than or equal to $C_\epsilon \left\lfloor \dfrac{n}{M} \right\rfloor \leq \frac{C_\epsilon}{M} \cdot n$.

Thus $\mathrm{Vol}(A) \approx x \cdot \dfrac{M-1}{M} \cdot n$ and $\mathrm{Vol}(B) \approx (1+\epsilon)\dfrac{n}{M}$.

NOTE: So one of the conclusions drawn from the 1st paper is that the large component of such a graph is based on $\tilde{d}$ as opposed to $d = \tilde{d}$ as in the $G_{n,p}$ model.

Paper 2

Let average distance be defined as $\overline{dist}(G) = \frac{1}{\binom{n}{2}} \sum_{u=v} dist(u,v)$.

Some conditions:

- $G(w)$ is <u>strongly sparse</u> which means

1. $\tilde{d}$ satisfies $0 < \log \tilde{d} \ll \log n$
2. For some constant $c > 0$ all but $o(n)$ vertices have $w_1 \geq c$ also $d = \frac{\sum w_i}{n} > 1$.

- $\mathbf{w}$ is admissable which means (1) and (2) hold from above and (3) $\exists U \subseteq V(G)$ satisfying

$$\begin{aligned} \mathrm{Vol}_2(U) &= (1+o(1))\mathrm{Vol}_2(G) \\ &\gg \frac{\mathrm{Vol}_3(U) \log \tilde{d} \log\log n}{\tilde{d} \log n} \end{aligned}$$

Theorem 1: If $\mathbf{w}$ is admissable the corresponding $G(w)$ has average distance $(1+o(1))\frac{\log n}{\log \tilde{d}}$ with probability approaching 1 as $n \to \infty$.