Lecture Notes Spring 2006 M690I: Extremal Graph Theory

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1 Introduction

For the following, all graphs are d-regular. They need not be simple graphs, but when computing the degree of a vertex, we only count loops once.

Let $S \subseteq V(G)$. The set $\partial S = E(S, \overline{S})$ is the number of edges going out of S. The isoperimetric constant of G is

$$h(G) = \min_{S:|S|\leq n/2} \frac{|\partial S|}{|S|} = \min_{S:|S|\leq n/2} \frac{|E(S,\overline{S})|}{|S|}.$$

There are other concepts of expansion, but the following is one of them:

A family of graphs $\{G_i\}$ is an **expander family** if

- it is d-regular of size n_i
- n_i does not grow too fast (typically, $n_{i+1} \leq n_i^2$)
- $\forall i$, $h(G_i) \geq \epsilon$

There are two examples of expander families which we will not prove.

Example 1 *We have* $V(G_m) = \mathbb{Z}_m \times \mathbb{Z}_m \setminus \{(0,0)\}$ *with*

$$(x,y) \sim \begin{cases} (x+y,y) \\ (x-y,y) \\ (x,x+y) \\ (x,x-y) \end{cases}$$

and all arithmetic is $\mod m$.

This is clearly 4-regular (if loops count once). Margulis (1973) [8] proved this was an expander and Gaber and Galil (1981) [6] proved that it is an ϵ-expander finding a specific $\epsilon > 0$.

Example 2 *We have $V_p = \mathbb{Z}_p$ for p prime with*

$$x \sim \begin{cases} x+1 \\ x-1 \\ \begin{cases} x^{-1} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases} \end{cases}$$

and all arithmetic is $\mod m$

This is 3-regular (if loops count once). Lubotsky, Philips and Sarnak (1988) [7] is an ϵ-expander.

2 Adjacency matrix

Let $\mathbf{A} = \mathbf{A}(G)$ be the adjacency matrix of G. In this case,

$$a_{ij} = \begin{cases} \#\{\text{edges between } i \text{ and } j\}, & i \neq j \\ \#\{\text{loops at } i\}\text{,i=j} \end{cases}$$

So, without loops, this is the usual adjacency matrix. Observe that $\mathbf{A}$ is symmetric and let it have an eigenbasis $\{\vec{v_0}, \vec{v_1}, \ldots, \vec{v_{n-1}}\}$ with corresponding eigenvalues $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$.

The following are known about the eigenvalues of $\mathbf{A}$:

- $\lambda_0 = d$
- G is connected iff $\lambda_0 > \lambda_1$
- G is bipartite iff $\lambda_0 = \lambda_{n-1}$

Proposition 1 *For all graphs G, let $\tilde{\deg}(v)$ denote the degree of v minus the number of loops and let $n = |V(G)|$, $e = |E(G)|$ and $\Delta(G) = \max_v \tilde{\deg}(v)$. Then, $\frac{2e}{n} \leq \lambda_0(G) \leq \Delta(G)$.*

Proof. To get the upper bound, let $\vec{v_0}$ be the eigenvector corresponding to λ_0. Let its entries be $\{x_k\}_{k=1}^n$ with x_j being the largest.

$$\lambda_0 \mathbf{v}_0 = (\mathbf{A}\mathbf{v}_0)_j = \sum_i a_{ij} x_i \leq x_j \sum_i a_{ij} = x_j \tilde{\deg}(v_j) \leq \Delta(G).$$

To get the lower bound, consider $\mathbf{1}^T\mathbf{A}\mathbf{1}$. We know that

$$\mathbf{1}^T\mathbf{A}\mathbf{1} \leq \lambda_0 \mathbf{1}^T\mathbf{1} = \lambda_0 n$$

because the expression $\frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ achieves both its maximum and minimum at eigenvectors.

But it is easy to see that

$$\mathbf{1}^T\mathbf{A}\mathbf{1} = \sum_v \tilde{\deg}(v) = 2e.$$

(Observe that the handshaking lemma gives that for any graph G, $\sum_{v \in V(G)} \deg(v) = 2e + \ell$, where ℓ denotes the number of loops. □

3 Using eigenvalues

The following theorem was proven in the "continuous case" by Bismut and Cheeger (I believe it's [3] and [4].)

Theorem 3.1 (Alon-Milman [2]) *Let G be a graph which is regular of degree d and has second-highest eigenvalue λ_1. Then,*

$$\frac{d-\lambda_1}{2} \leq h(G) \leq \sqrt{2d(d-\lambda_1)}$$

Proof. We prove the lower bound only via the following claim:

Claim 1 *Let G be a d-regular graph on n vertices and $S \subseteq V(G)$, then*

$$\left|E\left(S, \overline{S}\right)\right| \geq \frac{(d-\lambda_1)|S|\left|\overline{S}\right|}{n}.$$

Proof. Here we prove the claim. Of course the claim is trivial if $d = \lambda_1$. Now suppose $d > \lambda_1$. Observe that

$$\begin{aligned} \mathbf{x}^T(d\mathbf{I} - \mathbf{A})\mathbf{x} &= d\sum_i x_i^2 - 2\sum_{ij \in E(G)} x_i x_j \\ &= \sum_{ij \in E(G)} (x_i - x_j)^2. \end{aligned}$$

Observe further that if $\mathbf{1}$ is an eigenvector of $\mathbf{A}$ corresponding to d – hence an eigenvector of $d\mathbf{I} - \mathbf{A}$ corresponding to 0 – then the expression

$$\frac{\mathbf{x}^T(d\mathbf{I} - \mathbf{A})\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$$

where $\mathbf{x} \perp \mathbf{1}$ achieves its maximum and minimum when $\mathbf{x}$ is an eigenvector.

So, for $\mathbf{x} \perp \mathbf{1}$, $\mathbf{x}^T(d\mathbf{I} - \mathbf{A})\mathbf{x} \geq (d - \lambda_1)\mathbf{x}^T\mathbf{x}$. Let $s = |S|$ and

$$x_i = \begin{cases} s - n, & i \in S \\ s, & i \notin S \end{cases}$$

With this choice of $\mathbf{x}$,

$$\begin{aligned} n^2 \left|E\left(S, \overline{S}\right)\right| &= \sum_{ij \in E(G)} (x_i - x_j)^2 \\ &= \mathbf{x}^T(d\mathbf{I} - \mathbf{A})\mathbf{x} \\ &\geq (d - \lambda_1)\mathbf{x}^T\mathbf{x} \\ &= (d - \lambda_1)\sum_i x_i^2 \\ &= (d - \lambda_1)\left(s(s-n)^2 + (n-s)s^2\right) \\ &= (d - \lambda_1)s(n-s)n. \end{aligned}$$

Hence,

$$\left|E\left(S, \overline{S}\right)\right| \geq \frac{(d - \lambda_1)|S|\left|\overline{S}\right|}{n}$$

□

As a result,

$$\frac{\left|E\left(S, \overline{S}\right)\right|}{|S|} \geq \frac{(d - \lambda_1)\left|\overline{S}\right|}{n} \geq \frac{d - \lambda_1}{2}.$$

□

Theorem 3.2 (Alon-Boppana [1]) *Let $\{G_n\}$ be a family of d regular graphs so that G_n has n vertices and second-highest eigenvalue $\lambda_1(G_n)$. Then,*

$$\liminf_{n \to \infty} \lambda_1(G_n) \geq 2\sqrt{d-1}.$$

Definition 1 *A graph G is called **Ramanujan** iff*

$$\lambda_1(G) \leq 2\sqrt{d-1}.$$

Proposition 2 *Let G be a d-regular graph on n vertices with second-highest eigenvalue λ_1. Then*

$$\lambda_1 \geq \sqrt{d}\sqrt{\frac{n-d}{n-1}}.$$

Proof. Let A be the adjacency matrix of G.

$$\begin{aligned}\text{trace}(A^2) &= nd \\ &= \sum_i \lambda_i^2 \\ &\leq d^2 + (n-1)\lambda_1^2\end{aligned}$$

So, $(n-1)\lambda_1^2 + d^2 \geq nd$ and

$$\lambda_1^2 \geq \frac{nd - d^2}{n-1}.$$

□

Theorem 3.3 (Friedman [5]) *Let $G_{n,d}$ denote a random d-regular graph on n vertices. For all $\epsilon > 0$,*

$$\lim_{n\to\infty} \Pr\left(\lambda_1\left(G_{n,d}\right) \leq 2\sqrt{d-1} + \epsilon\right) = 1.$$

References

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