

Lemma 1 (Slicing Lemma) *Let us be given ϵ, α, d such that $\epsilon > 0$, $1 > \alpha > \epsilon$ and $d, 1 - d \geq \max\{2\epsilon, \epsilon/\alpha\}$. Let $A' \subseteq A$ with $|A'| \geq \alpha|A|$ and $B' \subseteq B$ with $|B'| \geq \alpha|B|$. Then (A', B') is ϵ' -regular with $\epsilon' = \max\{2\epsilon, \frac{\epsilon}{\alpha}\}$ and density in $(d - \epsilon, d + \epsilon)$.*

Proposition 1 (1) *Let $\epsilon > 0$ and $\epsilon'' > \epsilon$. Let (A, B) be an ϵ -regular pair with density d and $|A| = |B| = L$. Choose X uniformly at random from $\binom{A}{L_0}$ and Y uniformly from $\binom{B}{L_0}$, where $L_0 > \epsilon''L$.*

(Note: Define: $\binom{A}{L_0} = \{A' \subseteq A : |A'| = L_0\}$)

With high probability (i.e., probability approaching 1 as $L \rightarrow \infty$), each of the following holds:

1. $|\{y \in Y : |X \cap N(y)| \leq (d - \epsilon'')|X|\}| \leq \epsilon''|Y|$
2. $|\{y \in Y : |X \cap N(y)| \geq (d + \epsilon'')|X|\}| \leq \epsilon''|Y|$
3. $|\{(y_1, y_2) \in Y \times Y : |X \cap N(y_1) \cap N(y_2)| \leq (d - \epsilon'')^2|X|\}| \leq 2\epsilon''|Y|^2$
4. $|\{(y_1, y_2) \in Y \times Y : |X \cap N(y_1) \cap N(y_2)| \geq (d + \epsilon'')^2|X|\}| \leq 2\epsilon''|Y|^2$

These four statements also hold if the roles of X and Y are switched.

Proposition 2 (2) *If*

$$\frac{1}{|y|} \sum_{y \in Y} \left(\frac{|N(Y)|}{|X|} - d \right)^2 \leq \delta_1$$

and

$$\frac{1}{|Y|^2} \left| \left\{ (y_1, y_2) \in Y \times Y : \left| \frac{|N(y_1) \cap N(y_2)|}{|X|} - \frac{|N(y_1)| \cdot |N(y_2)|}{|X|^2} \right| \right\} \right| \leq \delta_3$$

then

$$|d(X', Y') - d| \leq \left(\delta_4 \frac{|X||Y|}{|X'||Y'|} \right)^{1/2}$$

with δ_i as defined previously.

Lemma 2 (Random Slicing) *Let $\epsilon' > 0, d \in (0, 1)$. There exists $\epsilon = \epsilon(\epsilon') > 0$ such that for all $c > \epsilon$, and all ϵ -regular pairs (A, B) where $|A| = |B| = L$ and $d(A, B) = d$, then choosing X uniformly at random from $\binom{A}{cL}$ and y uniformly at random from $\binom{B}{cL}$ gives, with high probability, an ϵ' -regular pair (X, Y) such that $|d(X, Y) - d| < \epsilon$*

Proof. (of Lemma)

Choose $\epsilon'', c > \epsilon'' > \epsilon$.

Choose $X \subseteq \binom{A}{cL}, Y \subseteq \binom{B}{cL}$ uniformly at random.

Proposition(1) gives properties 1,2,3,4. Assume the bad areas are as bad as possible, and there are as many as possible of them; assume the good ones are, well, good.

Note that in the first inequality to follow, the first term comes from the bad parts in one direction, the second from the bad parts in the other direction, and the third term comes

from the good parts.

Look at

$$\begin{aligned} \sum_{y \in Y} (|N(y) - d|X|)^2 &\leq \epsilon''|Y|(d|X|)^2 + \epsilon''|Y|[(1-d)|X|]^2 + |Y|(\epsilon''|X|)^2 \\ (2) &\leq |Y||X|^2 (d^2\epsilon' + (1-d)^2\epsilon'' + (\epsilon'')^2) \\ (3) &\leq |Y||X|^2(\epsilon'' + (\epsilon'')^2) \end{aligned}$$

In inequality (2), we rewrite. In inequality (3), note that the sum of the first and second terms are maximized when $d = 1, 0$, and minimized when $d = 1/2$.

So, let δ_1 in proposition 2 be greater than $\epsilon'' + (\epsilon'')^2$. Then Proposition 2 condition 1 holds.

Look to Proposition 2 condition 2.

(Aside for notation: $\#$ before a set means the same as $||$ around that set. This is done sometimes when there are multiple $||$'s).

$$\# \left\{ (y_1, y_2) : \left| |N(y_1) \cap N(y_2)| |X| - |N(y_1)| |N(y_2)| \right| \geq \delta_2 |X|^2 \right\} \leq \delta_3 |Y|^2$$

So, at most $4\epsilon''|Y|^2$ pairs $(y_1, y_2) \in Y \times Y$ fail to satisfy:

$$(d - \epsilon'')|X| \leq |N(y_1) \cap X|, |N(y_2) \cap X| \leq (d + \epsilon'')|X| \quad (1)$$

There are also at most $4\epsilon''|Y|^2$ pairs that fail to satisfy:

$$(d - \epsilon'')^2|X| \leq |N(y_1) \cap N(y_2) \cap X| \leq (d + \epsilon'')^2|X| \quad (2)$$

by condition 3,4 applied directly.

If both 1 and 2 hold, then

$$\begin{aligned} |X \cap N(y_1) \cap N(y_2)| &\leq (d + \epsilon'')^2|X| \\ &\leq [(d + \epsilon'')^2 - (d - \epsilon'')^2]|X| + \frac{|X \cap N(y_1)||X \cap N(y_2)|}{|X|} \\ &= 4\epsilon''d|X| + \frac{|X \cap N(y_1)||X \cap N(y_2)|}{|X|} \end{aligned}$$

So,

$$|X \cap N(y_1) \cap N(y_2)||X| - |X \cap N(y_1)||X \cap N(y_2)| \leq 4\epsilon''d|X|^2.$$

Also, now,

$$\begin{aligned} |X \cap N(y_1) \cap N(y_2)| &\geq (d - \epsilon'')^2|X| \\ &\geq [(d - \epsilon'')^2 - (d + \epsilon'')^2]|X| + \frac{|X \cap N(y_1)||X \cap N(y_2)|}{|X|} \end{aligned}$$

So,

$$|X \cap N(y_1) \cap N(y_2)| |X| - |X \cap N(y_1)| |X \cap N(y_2)| \geq -4\epsilon'' |X|^2$$

If (*) and (**) hold, then

$$\left| |X \cap N(y_1) \cap N(y_2)| |X| - |X \cap N(y_1)| |X \cap N(y_2)| \right| \leq 4\epsilon'' |X|^2$$

So let $\delta_2 = 4\epsilon''$ and $\delta_3 = 4\epsilon'' + 4\epsilon''$ (each $4\epsilon''$ from different places). Finally let $\delta_4 = \delta_1 + (\delta_2^2 + \delta_3)^{1/2}$ and with $(\delta_2^2 + \delta_3) = \epsilon''$ and $(\epsilon'')^{1/2} > \epsilon''$, this means δ_4 is $O(\sqrt{\epsilon''})$.

Therefore, $\forall X' \subseteq X, Y' \subseteq Y, |d(X', Y') - d| \leq \left(\delta_4 \frac{|X||Y|}{|X'||Y'|} \right)^{1/2}$.

In order for (X, Y) to be ϵ' -regular, (what we want in the lemma), we need $\left(\frac{\delta_4}{(\epsilon')^2} \right)^{1/2} < \epsilon'$.

Then you get $\delta_4^{1/4} < \epsilon'$.

If $\Omega((\epsilon'')^{1/8}) < \epsilon'$, then we have ϵ' -regularity for (X, Y) , i.e., there exists K such that $K(\epsilon'')^{1/8} < \epsilon'$

□