

Theorem of Alon and Yuster

THEOREM: (Alon and Yuster) Let $\alpha > 0$, Then there exists n_0 such that if $n \geq n_0$ and if $\delta(G) > (1/2 + \alpha)n$, then there exists a family of $\left\lfloor \frac{1-\alpha}{4}n \right\rfloor$ copies of C_4 (cycle of length 4).

PROOF: Apply RegLemDF to G . For now, choose $\varepsilon > 0$, $d > 0$ such that $\alpha \gg d \gg \varepsilon$. Consider the reduced graph G_r . There are ℓ clusters in the partition, so there are ℓ vertices in G_r (G_r is basically an auxiliary graph). v_1 and v_2 are adjacent in G_r if and only if in G' , (V_1, V_2) is ε -regular with density greater than d . We want $\delta(G_r) \geq (1 - 1/2)\ell$, so then G_r would have a matching of size $\left\lfloor \frac{\ell}{2} \right\rfloor$, by Hajnal-Szemerédi.

Now we ask ourselves, what is the minimum degree of G_r ? We know that for all vertices $v \in V(G)$ (remember that $|V(G)| = n$),

$$\deg_{G'}(v) \geq \deg_G(v) - (d + \varepsilon)n$$

Pick V_1 . We know

$$\begin{aligned} \sum_{v \in V_1} \deg_{G'}(v) &\geq \sum_{v \in V_1} \deg_G(v) \sum_{v \in V_1} (d - \varepsilon)n \\ &\geq L \binom{n}{2} - L(d + \varepsilon)n \\ &\geq \left(\frac{1}{2} + \alpha - d - \varepsilon \right) nL \end{aligned}$$

Now we count the edges incident to V_1 (in G'):

- edges inside V_1 (there are none!)
- edges in (V_1, V_0) (there are $\leq L|V_0|$ of these)
- edges in ε -regular pairs of density greater than d (amounts to $\leq \deg_{G_r}(v_1) \cdot L^2$)

Then,

$$\left(\frac{1}{2} + \alpha - d - \varepsilon \right) nL \leq \sum_{v \in V_1} \deg_{G'}(v) \leq \deg_{G_r}(v_1)L^2 + L|V_0|$$

Which implies

$$\left(\frac{1}{2} + \alpha - d - \varepsilon \right) nL \leq \deg_{G_r}(v_1)L + |V_0| \leq \deg_{G_r}(v_1)L + \varepsilon n$$

Thus,

$$\frac{\left(\frac{1}{2} + \alpha - d - 2\varepsilon \right) n}{L} \leq \deg_{G_r}(v_1)$$

Recall that $(1 - \varepsilon)n \leq \ell L \leq n$. This justifies the following statement, that

$$\left(\frac{1}{2} + \alpha - d - 2\varepsilon \right) \ell \leq \deg_{G_r}(v_1)$$

So, we need that $d + 2\varepsilon \leq \alpha$.

Thus there is a matching in the reduced graph by Hajnal-Szemerédi of size $\lfloor \frac{\ell}{2} \rfloor$. If ℓ is even, we don't do anything; if ℓ is odd, then throw the unmatched cluster into the leftover set V_0 . Now the leftover set is of size at most $2\varepsilon n$ (by **RegLemDF**).

Now we're going to relabel the clusters for ease in notation. We give them the labels A_i and B_i for $i = 1, 2, \dots, \lambda$, where A_i is matched with B_i in the matching guaranteed by Hajnal-Szemerédi, $\lambda = \lfloor \frac{\ell}{2} \rfloor$, and (A_i, B_i) is ε -regular with density $\geq d$.

We want to modify the pair (A_i, B_i) so that the pair is (ε, δ) -super-regular. Set $\delta = d - 2\varepsilon$. Form A'_i by removing at most $\lceil \varepsilon L \rceil + 1$ vertices, including all those with $\deg_{B_i}(a) < (d - \varepsilon)L$, where $a \in V(A_i)$. Then we'll have that (A'_i, B_i) is $(\varepsilon, d - 2\varepsilon)$ -super-regular.

So, we note at this point that

$$\begin{aligned} |V_0| &\leq 2\varepsilon n + 2\lambda(\lceil \varepsilon L \rceil + 1) \\ &\leq 2\varepsilon n + \ell(\lceil \varepsilon L \rceil + 1) \\ &\leq 2\varepsilon n + \ell(\varepsilon L + 2) \\ &\leq 2\varepsilon n + \varepsilon \ell L + 2\ell \\ &\leq 3\varepsilon n + 2\ell \\ &\leq 3\varepsilon n + 2M(\varepsilon), \end{aligned}$$

where $M(\varepsilon)$ comes from **RegLemDF**. If each A'_i, B'_i can be covered by vertex-disjoint copies of C_4 , then we have covered all but $3\varepsilon n + 2M(\varepsilon)$ vertices by copies of C_4 .

If $\frac{\alpha}{4}n \geq 3\varepsilon n + M(\varepsilon)$, then we have a family of $\lfloor \frac{1 - \alpha}{4}n \rfloor$ vertex disjoint copies of C_4 .

Note: We need $\alpha \geq 12\varepsilon + \frac{8M(\varepsilon)}{n}$.

If $n_0 = M(\varepsilon)$, this is easy.

Finally, we consider the pair (A'_i, B'_i) , where $|A_i| = |B_i| = 2L'$ (since they are sets of even size). Then we have

$$2L' \geq L - \lceil \varepsilon L \rceil - 1 \geq (1 - 2\varepsilon)L$$

Recall that the pair is $(\varepsilon, d - 2\varepsilon)$ -super-regular, so we can apply the Blow-U Lemma. Recall that the Blow-Up Lemma requires

$$\begin{aligned} H &= \{L' \text{ vertex-disjoint copies of } C_4\}, \\ \delta &= d - 2\varepsilon, \end{aligned}$$

and we're given $\Delta = \Delta(H) = 2$, that $R = K_2$, and that $R(2L') = K_{2L', 2L'}$.

Certainly H is embeddable in $R(t)$. So, given $d - 2\varepsilon$, there is an ε' such that G , being the $(\varepsilon', d - 2\varepsilon)$ -super-regular approximation of $R(t)$, contains H as a subgraph. (At this point, we note that ε' is the “ ε ” we get from the Blow-Up Lemma.)

Now we have to go through the horrendous details of computing the inequalities we need for all of the parameters so as to make this business work smoothly. We won't normally be

computing these, and we really just need to go through it once to prove to ourselves (and others) that it can actually be done.

We choose them in this manner:

$$\begin{aligned}\frac{8M(\varepsilon)}{n_0} &< \varepsilon \\ \alpha &> 13\varepsilon \\ \alpha &> d + 2\varepsilon \\ d &> 2\varepsilon \\ \varepsilon &< 1/2\end{aligned}$$

Those were a bunch of inequalities that have to hold in order for this to work. So our choices go in this order:

$$d < \frac{2\alpha}{3} \tag{1}$$

$$\varepsilon = \min \left\{ \frac{\alpha}{13}, \frac{d}{4}, \underbrace{\varepsilon'(d/2, 2)}_{\text{B-U Lemma}} \right\} \tag{2}$$

$$n_0 > \frac{8M(\varepsilon)}{\varepsilon} \tag{3}$$

A simple check will show that all the inequalities above hold indeed. ■

So what did we even do to prove this? If we analyze the structure of the proof, we see that it wasn't too out of control:

- First, smash the graph into pieces with `RegLemDF`
- Second, ensure that G_r has large enough minimum degree
- Then use Hajnal-Szemerédi to ensure a matching in the auxiliary graph G_r
- Fourthly, slice, to get your (ε, δ) -super-regular matching
- Lastly, apply the Blow-Up Lemma

Not too shabby! Now we can think about what we would do if we wanted copies of K_{m_1, m_2} . The proof would change only slightly; that is, we would have to ensure that our clusters are divisible by $m_1 + m_2$. (Recall that we ensured that our clusters were divisible by 2, since we were interested in the complete bipartite graph on partite sets of size one; that is, just a single edge between *two* vertices. Then the analogue of the K_{m_1, m_2} situation becomes apparent.)