690I Scribe Notes for Feb 28 2006

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0.1 scriven

Recall Erdős-Stone:

For any integers $p \geq 2$, $t \geq 1$,

$$ex(n, K_k(t, ...t)) = \left(1 - \frac{1}{p-1} * \binom{n}{2}\right) + o(n^2\right)$$

where ex(n,G) is the maximum number of edges in an order n graph with no subgraph isomorphic to G

Erdős-Simonovits:

Let L be a family of graphs, $p = min\{\chi(l)\}$ 1, $l \in L$. Let $ex(n, L) =$ the maximum number of edges in an order n graph with no subgraphs isomorphic to any member $l \in L$. Then $ex(n, L) = (1 - \frac{1}{p-1})\binom{n}{2} + o(n^2)$.

The proof is when

$$\max_{l \in L}(|V(l)|) \leq K$$

.

If you have a $K_p(K, ...K) \supseteq l$ in the original.

Let $\epsilon = \left(\frac{\beta}{6}\right)^h$, $h = |V(H)|$.

Then $e(G_n) > \left(1 - \frac{1}{p-1} + \beta\right)\binom{n}{2}$

$$\Longrightarrow \|H \to G_n\| > \left(\frac{\epsilon n}{M(\epsilon)}\right)^h$$

Recall Hajnal-Szemerédi
If $\delta(G_n) \geq \left(1 - \frac{1}{r}\right) n$, then there exists a subgraph which consists of $\lfloor \frac{n}{r} \rfloor$ vertex disjoint copies of K_r for all n.

Let C_4 be the cycle on 4 vertices.

If $\delta(G_n) \geq \frac{3}{4}n$, then $\exists \lfloor \frac{n}{r} \rfloor$ vertex disjoint copies of K_4, hence C_4.

Equivalent to Hanjal-Szemerédi if $\Delta(G_n) < \frac{n}{r}$ and $r|n$, then there exists a proper coloring of G_n where every color class is of size exactly r.

Theorem 1 (Alon-Yuster(1992))**.** *$\forall_{\alpha > 0}$ and graph H, $\exists_{n_0}$ such that in $n \geq n_0$, $\delta(G_n) > (1 - \frac{1}{\chi(H)} + \alpha)n$ then there is a family of $\lfloor \frac{1-\alpha}{|V(H)|} n \rfloor$ vertex-disjoint copies of H in G_n.*

If $\alpha(G_n) > (\frac{1}{2}+\alpha)n$ then there is a family of $\lfloor\frac{(1-\alpha)n}{4}\rfloor$ vertex-disjoint copies of C_4 (for $n \geq n_0$).

Conjecture 1. *$\forall H, \exists K$ such that if $\delta(G_n) > (1-\frac{1}{\chi(H)})n$ then G_n has a family of vertex-disjoint copies of H that we use up all but K vertices. ($k \neq 0$ even if $V(H)$ divides n)*

Let $H = K_2, \delta(G_n) \geq \frac{n}{2}$

Dirac says $G_n \supseteq C_n$ a cycle on n vertices.
If n is even $C_n \supseteq$ a perfect matching.
If n is odd $C_n \supseteq$ a matching of size $\lfloor\frac{n}{2}\rfloor$.

If $H = K_1$ then $K = 0$, if $H = K_2$ then $K = 1$.

$K \neq 0$ is shown by Alon-Yuster[1].

Tripartite $G = (V_1, V_2, V_3)|V_1| = |V_2| = |V_3| = N$.

If each bipartite graph (v_i, v_j) has minimum degree $\leq \frac{2N}{3h} + 2h - 1$, then there exists $\lfloor\frac{N}{h}\rfloor$ copies of $K_{h,h,h}$.

Definition 1 ((ϵ, δ)-super-regularity)**.** *Given a pair (A, B) we say that (A, B) is $(\epsilon - \delta)$-super-regular if $\forall X \subseteq A, \forall Y \subseteq B$ satisfying $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have $e(x, y) > \delta|X||Y|$ and*
$deg(a) > \delta|B|\forall_{a\in A}$
$deg(b) > \delta|A|\forall_{b\in B}$

Theorem 2. *Let $\epsilon < \frac{1}{2}, d > 2\epsilon$. Let (A, B) be an ϵ-regular pair with density d, $|A| = |B| = L$.*

Then $\exists A' \subseteq A$, $B' \subseteq B$ such that (A', B') is $(\epsilon, d - 2\epsilon)$ super-regular, $|A'|, |B'| \geq (1-\epsilon)L$
$d(A', B') \in (d-\epsilon, d+\epsilon)$

Lemma 1 (Blow-up lemma by Komlós-Sárközy-Szemerédi(1994))**.** *Given a graph R of order r and positive parameters, δ, Δ, there exists $\epsilon > 0$ such that the following holds.*

Let $n_1, ..., n_r$ be arbitrary positive integers and replace the vertices of R with pairwise disjoint sets $V_1, ..., V_r$ of sizes $n_1, ..., n_0$ (blowing up)

$R(n_1, ..., n_r)$ is obtained by replacing each edge $(v_i, v_j) \in E(R)$ with the complete bipartite graph between V_i and V_j.

G is obtained by replacing each edge $(v_i, v_j) \in E(R)$ with an (ϵ, δ) super regular pair.

If a graph H, $\Delta(H) \leq \Delta($max degree$)$ is embeddable into $R(n_1, ..., n_r)$ then it is embeddable into G.

For example let δ be given, then there exists $\epsilon > 0$ such that every (ϵ, δ)-super-regular pair (with both sets the same size) is Hamiltonian.

$R = K_2$
$R(L, L)$ is Hamiltonian
$G = (A, B)$, $|A| = |B| = L$, (ϵ, δ)-super-regular

Every ϵ-regular pair $|A| = |B| = L$ of density $\geq \delta + 2\epsilon$, $(\epsilon < \frac{1}{2})$, has a cycle of length $\geq 2(1 - \epsilon)L$.

Bibliography

[1] Noga Alon and Yuster. Almost h-factors in dense graphs. *Graphs and Combinatorics*, 8:95–102, 1992.