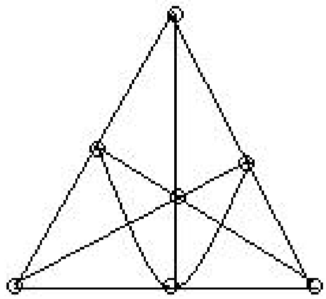
Early Applications

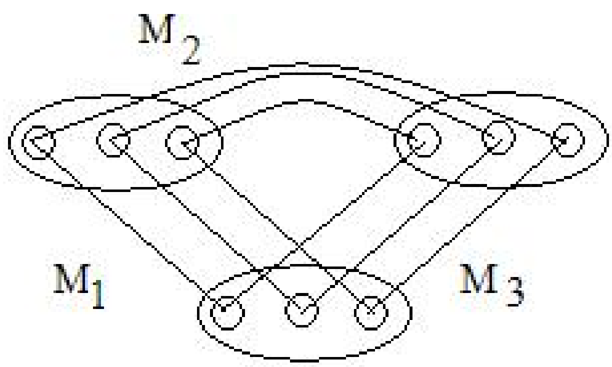
Theorem 0.1 *Let H_n be a 3-uniform Hypergraph. if H_n contains no (6,3) configuration, then $e(H_n) = o(n^2)$.*

A **3-uniform hypergraph** is a set system where all sets are of size 3.
A (6,3)-configuration is 6 vertices with at least 3 triangles (i.e. Hyperedges, 3-sets). Note that cycles are ill-defined when dealing with hypergraphs.

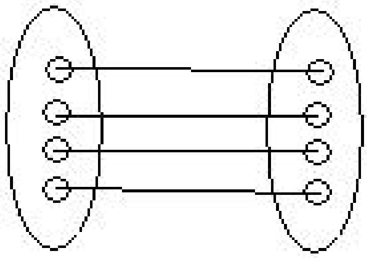
Example: Graph of Fano plane:



Lemma 1 *If G_n is the union of n induced matchings, then $e(G_n) = o(n^2)$. For G_n, $\exists m_1, \ldots, m_k$ such that each edge is in one of these.*
The only edges in G connecting the vertices of M are a part of M_i.

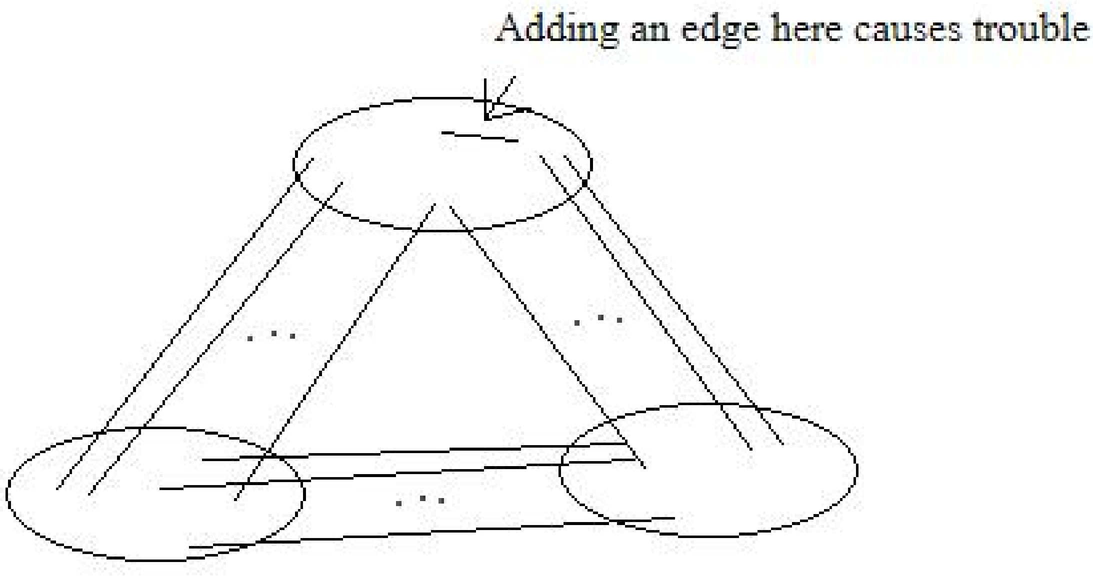


The Lemma was proven with `RegLem` , which proves the theorem.
We partition the edge set into $M_1, \ldots, M_n$. The vertices covered by M_i form an induced matching. If M_1 is



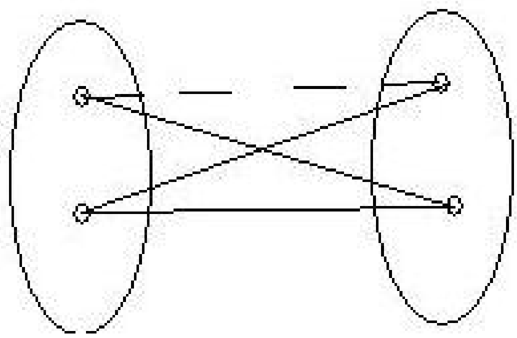
then $\nexists$ other edges in this edge set. $H \subseteq G$ edges/nonedges same in H and G when you look at the homomorphism $\psi : H \to G$ (edge preserving).

Theorem 0.2 *(Turán)*
If G_n has no copy of K_4, then $e(G_n) \leq (1 - \frac{1}{3})\binom{n^2}{2} = \frac{n^2}{3}$.
Extremal Configuration:



Proposition 1 *(Ramsey-Turán-type (by Szemerédi))*
Let G_n contain no K_n and $\alpha(G_n) = o(n)$. Then $e(G_n) < \frac{n^2}{8} + o(n^2)$.
Bollobás-Erdős gave a construction with no K_4 and $e(G_n) > \frac{n^2}{8}$.

A **2-diameter critical graph** is one which has diameter 2 but the deletion of any edge increases the diameter to 3 or more.
$K_{\lceil n/2 \rceil, \lfloor n/2 \rfloor}$ is a 2-diameter critical graph. Deletion of any edge means a distance of 3.



Conjecture 1 *(Murty, Simon - 1974)*
If G_n is a 2-diameter critical grpah, then $e(G_n) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. This seems clear, but it is still open. It's "solved" in an asymptotic sense, as follows.

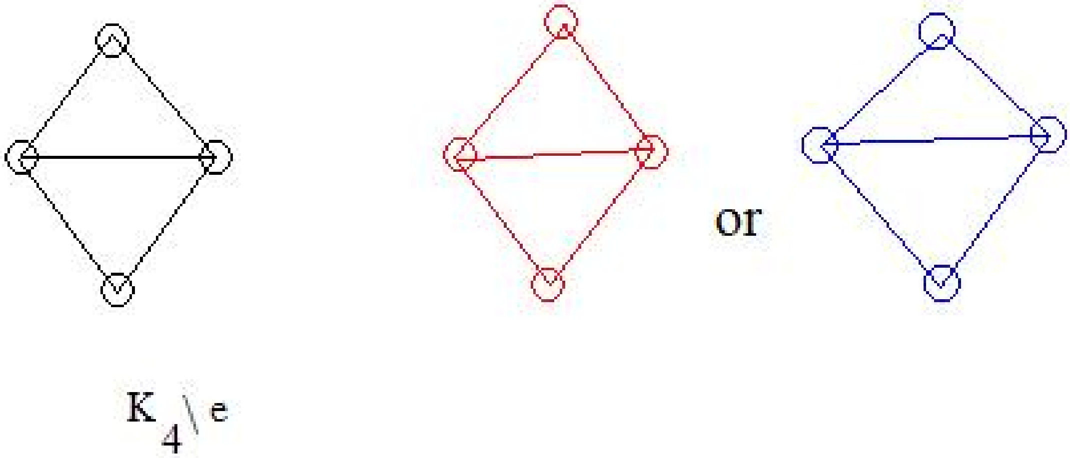
Theorem 0.3 *(Füredi - 1992)*
There is an integer n_0 such that if $n \geq n_0$ and G_n is a 2-diameter critical graph, then $e(G_n) \leq \left\lfloor \frac{n^2}{4} \right\rfloor$. This seems ok. It's at least as good as the Ramsey-Turán-type result (by Szemerédi) above.

A graph $G = (v, e)$ has the (γ, δ, σ) **property** if $\forall S \subseteq V$ with $|S| > \delta|V|$, the induced subgraph $G[S]$ satisfies $(\sigma - \delta)\binom{|S|}{2} \leq e(G[S]) \leq (\sigma + \delta)\binom{|S|}{2}$. This definition is due to Rödl.

Theorem 0.4 *(Rödl - 1986) For every positive integer k and every $\sigma > 0, \delta > 0$ such that $\delta < \sigma < 1 - \delta$, there exists a $\gamma = \gamma(k, \sigma, \delta)$ and a positive integer n_0 such that every graph G_n with $n \geq n_0$ vertices satisfying the (γ, δ, σ) property contains all graphs with k as induced subgraph.*

Corollary 1 *For every graph L, there exists a graph H (thus making this un-Ramsey-like) such that for any 2-coloring of the edges of H, H contains an induced monochromatic subgraph isomorphic to L. (Mostly a Ramsey-like statement except for the noted portion). In particular, L can be made a family of graphs $\mathcal{L}$. (apply corollary to the disjoint union of elements of $\mathcal{L}$).*

Induced means the edges of L are red, for example, and the non-edges are absent.
Suppose $L = K_n \setminus \{e\}$. Then there exists H such that no matter how we red/blue color H, there exists either:



HOMEWORK: Let G be a graph where vertices are elements of $\binom{[2k+1]}{k}$ and 2 vertices are adjacent iff they have nonempty intersection. Prove or disprove G is 2-diameter critical. ([2k+1] is the set of integers from 1 up to 2k+1).

Conjecture 2 *(Erdős, Sós - 1965)*
Every graph on n vertices with at least $\frac{k-1}{2}n$ edges contains as subgraphs all trees with k edges (i.e. $k+1$ vertices).

Theorem 0.5 *(Ajtai, Komlós, Szemerédi - 1991) For every $\epsilon > 0$, there exists a positive integer k_0 such that for $k > k_0$, every graph with average degree at least $(1+\epsilon)k$ contains, as subgraphs, all trees with k edges.*
If G_n has $\frac{k-1}{2}n$ edges, the average degree is $2(\frac{k-1}{2}n)/n = k-1$.
The proof has two parts: `RegLem` *, if you have a dense section (the number of edges is quadratic) or:*
Sparse version of `RegLem` *, since with* `RegLem` *, if $e(G_n) = o(n^2)$, you can say nothing.*

Conjecture 3 *(Loebl)*
If G_n is a graph such that at least 1/2 of the vertices have degree at least $n/2$, then G contains as subgraphs all trees with at most $n/2$ edges.
(For $n \geq n_0$ and a factor of $1 + \epsilon$ by Zhao).

Path and star are easy cases here, but the more "average" trees are harder (for example, binary-splitting trees).