# Lecture Notes Spring 2006 M690I: Extremal Graph Theory

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## 1 Regularity Lemma, Degree Form [RegLemDF]

For every $\varepsilon > 0$, there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number then there is a partition of the vertex set $V$ into $\ell + 1$ clusters $V_0$, $V_1$, ..., $V_\ell$ and there is a subgraph $G' \subseteq G$ with the following properties:

- $\ell \leq M$ (no lower bound)
- $|V_0| \leq \varepsilon|V| = \varepsilon n$
- all clusters $V_i$, $1 \leq i \leq \ell$ are the same size $L \leq \varepsilon n$
- $\deg_{G'} v > \deg_G v - (d + \varepsilon)n$
- $e(G'[V_i]) = 0$
- all pairs of clusters are $\varepsilon$-regular with density either 0 or at least $d$

Proof: Homework. Hint: Apply RegLem with $(\varepsilon')^2$ adn $m = \lceil (\varepsilon')^{-1} \rceil$.

1. Remove all edges inside a cluster (not leftover).
2. Remove all clusters in at least $\varepsilon' \ell^2$ irregular pairs, and place all such vertices in the leftover set.
3. Remove all edges within remaining irregular pairs.
4. Remove all vertices $v \in V_i$ such that $\deg_{U_i} v \geq (d + \varepsilon')n$, with

$$U_i = \bigcup_{\substack{j=1 \\ j \neq i}}^{\ell} \{V_j : d(V_i, V_j) \leq d + \varepsilon'\}.$$

[There are not many; remove the same number from each cluster to get them the same size.] Put these in the leftover set.

5. Remove all edges in sparse pairs (pairs with density $\leq d$].

The resulting partition satisfies the theorem. The removed edges result in the subgraph $G'$. The leftover set is now of size $\leq 3\varepsilon' n$ (check this) and $\deg_{G'} v > \deg_G v - (d + 3\varepsilon')n$.

# 2 Key Lemma

Given a graph $R$ (small), replace each vertex in $R$ with a set of $t$ vertices. Edges in $R$ result in complete bipartite graphs $K_{t,t}$. No edges between the $t$-vertex sets.

**Key Lemma** Let $d > \varepsilon \geq 0$. Given $R$ and a positive integer $L$, let us construct $G$, replacing each vertex of $R$ by $L$ vertices and replacing the edges of $R$ with $\varepsilon$-regular pairs of density at least $d$. Let $N$ be a subgraph of $R(t)$, $h = |V(H)|$, and $\Delta(H) = \Delta$ (max deg), $\Delta > 0$, let $\delta = d - \varepsilon$ and $\varepsilon_0 = \dfrac{\delta\Delta}{2+\Delta}$. If $\varepsilon < \varepsilon_0$ and $t - 1 \leq \varepsilon_0 L$, then $H$ is a subgraph of $G$. Moreover, $\|H \to G\| > (\varepsilon_0 L)^h$, where $H \to G$ is the set of the embeddings (copies) of $H$ in $G$.

Proof: *If $t - 1 \leq (\delta^\Delta - \Delta\varepsilon)L$, then $\|H \to G\| > \big((\delta^\Delta - \Delta\varepsilon)L - (t-1)\big)^h$.* Define $V(H) = \{v_1, v_2, \ldots, v_n\}$. Assign $v_i$ to a cluster of $R(t)$ in whatever embedding works. For $v_j$, define $C_{i,j}$, $i < j$ to be the sets from which $v_j$ can be chosen at stage $i$. $C_{0,j} \supseteq C_{1,j} \supseteq C_{2,j} \supseteq \cdots \supseteq C_{j-1,j}$.

At stage $i \geq 1$: (1) Let $v_i \in C_{i-1,i}$ such that $\deg_{C_{i-1,j}} v_i > \delta |C_{i-1,j}|$ for all $j > i$ with $v_i \sim_H v_j$. (2) Update $C_{i,j} \stackrel{\text{def}}{=} C_{i-1,j} \cap N(v_i)$ if $v_i \sim_H v_j$, else $C_{i,j} = C_{i-1,j}$ For $j > i$, let $d_{i,j} = \big|\big\{k \in \{1, \ldots, i\} : v_k \sim_H v_j\big\}\big|$. So,

$$|C_{i,j}| > \begin{cases} \delta^{d_{i,j}} L, & \text{if } d_{i,j} > 0 \\ L, & \text{if } d_{i,j} = 0 \end{cases}$$

So, $|C_{i,j}| > \delta^{d_{i,j}} L \geq \delta^\Delta L > \varepsilon L$. (Choose $\varepsilon < \delta^\Delta$, with $\delta = d - \varepsilon$ and $\Delta =$ max deg in $H$; or choose $\varepsilon = \max\{d/2, (d/2)^\Delta\}$.)

When we choose $V_i$, at most $\Delta\varepsilon L$ choices are involved by $\varepsilon$-regularity. By $\varepsilon$-regularity, if $X' \subseteq X$ with $|X'| \geq \varepsilon|X|$ and $Y' \subseteq Y$ with $|Y'| \geq \varepsilon|Y|$, there exists $y \in Y'$ such that $\deg_{X'} y \geq (d - \varepsilon)|X'|$. Moreover,

$$|\{y \in Y : \deg_{X'} y < (d - \varepsilon)|X'|\}| < \varepsilon |Y|.$$

The number of bad to any other cluster is less than or equal to $(\deg_H v_i)\varepsilon L$.

So, at stage $i$, there are at least $(\delta^\Delta - \varepsilon\Delta)L$ valid choices for $V_i$.

$|C_{i-1,i}| \geq \delta^\Delta L$ and the number of bad vertices $\leq \Delta\varepsilon L$.

Use the counting principle.

The $t-1$ comes from the fact that we cannot choose $v_j = v_i$, $j > i$, even if they are in the same cluster. So, given $v_1, \ldots, v_{i-1}$, there exists at least $(\delta^\Delta - \varepsilon\Delta)L - (t-1)$ choices from $v_i$, and hence

$$\|H \to G\| > \left[(\delta^\Delta - \varepsilon\Delta)L - (t-1)\right]^h.$$

□

Thus, $\varepsilon_0 = \dfrac{\delta\Delta}{2+\Delta}$, $t - 1 \leq \varepsilon_0 L \quad \Rightarrow \quad \|H \to G\| > (\varepsilon_0 L)^h$.

**# of copies of $H$** Let $H$ be a graph with $h$ vertices and chromatic number $\chi(H) = p$. Let $\beta > 0$ and $\varepsilon = \left(\dfrac{\beta}{6}\right)^h$. If $n$ is large enough and $G_n$ a graph on $n$ vertices with $e(G_n) \geq \left(1 - \dfrac{1}{\beta - 1} + \beta\right)\dfrac{n^2}{2}$, then $\|H \to G_n\| \geq \left(\dfrac{\varepsilon n}{M(\varepsilon)}\right)^h$, where $M(\varepsilon)$ comes from RegLemDF.

PROOF: Homework. Hint: Follow Erdős-Stone by applying RegLemDF to $G_n$ and find $K_p$ in reduced graph, then apply Key Lemma.

**Covering copies of $H$** For every $\beta > 0$ and $H$ sample graph with $h$ vertices, there exists $\gamma = \gamma(\beta, H) > 0$ such that if $G_n$ is a graph with at most $\gamma n^h$ copies of $H$, then by deleting at most $\beta n^2$ edges, $G_n$ can be made $H$-free.