PROPOSITION: Let (A, B) be ε-regular with density d. If (A', B') has $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$, then (A', B') is an ε'-regular pair, with $\varepsilon' = \max\left\{2\varepsilon, \frac{|A|}{|A'|}\varepsilon, \frac{|B|}{|B'|}\varepsilon\right\}$.

PROOF: Let $A'' \subseteq A'$ and $B'' \subseteq B'$ with $|A''| \geq \varepsilon'|A'| \geq \varepsilon|A|$ and $|B''| \geq \varepsilon'|B'| \geq \varepsilon|B|$. Hence,

$$|d(A'', B'') - d(A, B)| \leq \varepsilon.$$

Then by the triangle inequality,

$$|d(A'', B'') - d(A', B')| \leq \varepsilon + |d(A', B') - d(A, B)| \leq 2\varepsilon.$$

■

LEMMA: Let t and p be fixed integers. Then $\forall\ d \in (0, 1)$, $\exists\ \varepsilon_0 = \varepsilon_0(d, t, p) = \varepsilon_0(d)$, and $L_0 = L_0(d, t, p) = L_0(d)$ such that whenever $V_1, V_2, \ldots, V_p$ is pairwise ε-regular with density at least d and $|V_1| = |V_2| = \ldots = |V_p| = L \geq L_0$, it contains a copy of $K(\underbrace{t, t, \ldots, t}_{p \text{ times}})$. [Recall that $K(\underbrace{t, t, \ldots, t}_{p \text{ times}})$ is the complete p-partite graph with t vertices in each partite set.]

PROOF: We proceed by induction on p.

Base: $p = 2$. ε-regular with density $d \ll \varepsilon$, so we just use the intersection property. $(d - \varepsilon)^{t-1}L \geq \varepsilon L$ and $t\varepsilon L^t \geq 1$. ✓

Now, suppose the statement is true for p, we have $\varepsilon_0 = \varepsilon_0(d, t, p)$ and $L_0 = L_0(d, t, p)$, so we need to prove it for $(p + 1)$, and get $\varepsilon_0(d, t, p + 1)$ and $L_0(d, t, p + 1)$.

Recall the intersection property. If $(d - \varepsilon)^{t-1}L \geq \varepsilon L$ then

$$\left|\left\{\overline{a} \in A^t : |\bigcap_{i=1}^{t} N(a_i)| \leq (d - \varepsilon)^t L\right\}\right| \leq t\varepsilon L^t.$$

Then

$$\left|\left\{\overline{a} \in V_{p+1}^t : |\bigcap N(a_i)| \leq (d - \varepsilon)^t L \text{ for any } V_i \text{ with } 1 \leq i \leq p\right\}\right| \leq tp\varepsilon L^t.$$

If $tp\varepsilon < 1$, $\exists\ t$ vertices with neighborhoods of size at least $(d - \varepsilon)^t L$ in $V_1, \ldots, V_p$.

Let $V_i' \subseteq V_i$, with $|V_i'| \geq \lceil (d - \varepsilon)^t L \rceil$. Note that for $i \neq j$, (V_i', V_j') is $\max\left\{2\varepsilon, \frac{|V_i|}{|V_i'|}\varepsilon, \frac{|V_j|}{|V_j'|}\varepsilon\right\}$-regular of density greater than $(d - \varepsilon)$.

Also note that if $|V_i'| \geq \varepsilon|V_i|$ and $|V_j'| \geq \varepsilon|V_j|$, then

$$\max\left\{2\varepsilon, \frac{|V_i|}{|V_i'|}\varepsilon, \frac{|V_j|}{|V_j'|}\varepsilon\right\} = \max\left\{2\varepsilon, \frac{\varepsilon}{(d - \varepsilon)^t)}\right\}.$$

We need $(d - \varepsilon)^t \geq \varepsilon$. If $2\varepsilon < \varepsilon_0(d - \varepsilon, t, p)$ and $\frac{\varepsilon}{(d - \varepsilon)^t} < \varepsilon_0(d - \varepsilon, t, p)$ and $(d - \varepsilon)^t < L_0(d - \varepsilon, t, p)$, then we can apply the inductive hypothesis, and see that $(V_1', \ldots, V_p')$ contains $K(\underbrace{t, t, \ldots, t}_{p})$, giving a copy of $K(\underbrace{t, t, \ldots, t}_{p+1})$ in the original graph.

To see this, set

$$\varepsilon_0(d,t,p+1) = \min\left\{1/2, (d-\varepsilon)^t\right\} \cdot \varepsilon_0(d,t,p)$$

and

$$L_0(d,t,p+1) = \frac{1}{(d-\varepsilon)^t)} L_0(d-\varepsilon,t,p).$$

■

The next theorem is due to Erdős and Stone in 1946. A proof of this can be found in *Combinatorial Geometry* by János Pach and Pankaj K. Agarwal on page 126. The book was published by John Wiley & Sons, Inc., in 1995.

THEOREM: (Erdős-Stone, 1946.) Let $\varepsilon > 0$. There exists $n_0 = n_0(p,t,\varepsilon)$ such that if a graph G has at least $n \geq n_0$ vertices and $e(G) \geq \left(1 - \frac{1}{p-1} + \varepsilon\right)\frac{n_0^2}{2}$, then there exists a copy of $K(\underbrace{t,t,\ldots,t}_{p})$ in G.

Does this sound familiar? It should! Think Turán bound. If $t = 1$, this is Turán's theorem:

$$\textit{If } |e(G)| \geq \left(1 - \frac{1}{p-1}\right)\binom{n}{2} + O(n), \textit{ then } G \textit{ contains } K_p.$$

PROOF: We use the reduced graph. For some d, define G_r to be the reduced graph on clusters $V_1,\ldots,V_\ell$, with two vertices v_i and v_j being adjacent in G_r whenever (V_i,V_j) is ε-regular with density greater than d.

Apply RegLem to G with ε' (which we will choose later) and $m = \lceil(\varepsilon')^{-1}\rceil$. The number of edges that are not in ε'-regular pairs is $\leq \left(5 \cdot \frac{\varepsilon'}{2}\right) n^2 \leq 3\varepsilon' n^2$.

Now, for some d (again, chosen later), what is the number of edges in the ε-regular pairs of density less than or equal to d? Let this number be x, and let ℓ be the number of non-exceptional clusters, where L is the size of each of those clusters.

So,

$$\begin{aligned} e(G) &\leq 3\varepsilon' n^2 + xdL^2 + \left(\binom{\ell}{2} - x\right)L^2 \\ &= 3\varepsilon' n^2 + \binom{\ell}{2}L^2 - (1-d)xL^2 \\ &\leq 3\varepsilon' n^2 + \frac{\ell^2 L^2}{2} - (1-d)xL^2 \\ &\leq 3\varepsilon' n^2 + \frac{n^2}{2} - (1-d)xL^2 \end{aligned}$$

Then,

$$\frac{n^2}{2} - \frac{1}{p-1}\cdot\frac{n^2}{2} + \varepsilon\frac{n^2}{2} \leq e(G) \leq \frac{n^2}{2} + 3\varepsilon' n^2 - (1-d)xL^2$$

Thus,

$$(1-d)xL^2 \leq \frac{1}{p-1} \cdot \frac{n^2}{2} - \left(\frac{\varepsilon}{2} - 3\varepsilon'\right) n^2$$

Which implies that

$$x \leq \frac{1}{2(p-1)(1-d)} \left(\frac{n}{L}\right)^2 - \left(\frac{\varepsilon}{2} - 3\varepsilon'\right) \left(\frac{n}{L}\right)^2$$

Since $(1-\varepsilon')n \leq \ell L \leq n$, we have

$$x \leq \frac{1}{2(p-1)(1-d)} \cdot \frac{\ell^2}{(1-\varepsilon')^2} - \left(\frac{\varepsilon}{2} - 3\varepsilon'\right) \ell^2$$

By Turán's theorem, if $x \leq \dfrac{\ell^2}{2(p-1)} - c\ell$ for some constant c, then G_r has a copy of K_p. Hence, if $\varepsilon' \ll d \ll \varepsilon$, we'll have it. So,

$$\left[\frac{1}{2(p-1)(1-d)(1-\varepsilon')^2} - \left(\frac{\varepsilon}{2} - 3\varepsilon'\right)\right] \ell^2 \leq \left[\frac{1}{2(p-1)} - \frac{c_p}{\ell}\right] \ell^2$$

Thus,

$$\frac{1}{2(p-1)(1-d)(1-\varepsilon')^2} + 3\varepsilon' + \frac{c_p}{\ell} - \frac{1}{2(p-1)} \leq \frac{\varepsilon}{2} (*)$$

Note:

$$\frac{c_p}{\ell} \leq \frac{c_p}{m} = \frac{c_p}{(\varepsilon')^{-1}} = \varepsilon' c_p$$

(These calculations are left to the non-scribes [that means you!] if interested.)

Use the previous lemma with L as given in RegLem and ε' and d. $\varepsilon' \ll d$ by conditions of the lemma. If d is small compared to ε, then $(*)$ holds, so we get a copy of $K(\underbrace{t, t, \ldots, t}_{p})$ in G itself. ■

Given p. Works for some $t \leq \dfrac{\varepsilon \log n}{2^{p+1}(p-1)!}$ and $n_0 = n_0(p, \varepsilon) = \max\{\lceil 3/\varepsilon \rceil, 100\}$. (This result is due to Bollobás and Erdős, in 1973.)

What do we get? We start off with a large graph, apply RegLem , thus focing a clique in the reduced graph, and forces a small subgraph. We end up getting anything we want to get between clusters!

Corollary: If $|V(H)| \leq t$ and $\chi(H) = p$, then there exists and n_0 such that if $n \geq n_0$ and if $e(G) > \left(1 - \dfrac{1}{(p-1)} + \varepsilon\right) \dfrac{n^2}{2}$, then there is a copy of H in G (n is the number of vertices in G).