Definition: Let $(A, B; E)$ be a bipartite graph with density d. We say that $(A, B; E)$ is ϵ-regular if $\forall X \subseteq A, \forall Y \subseteq B$ with $|X| \geq \epsilon|A|, |Y| \geq \epsilon|B|$ we have $|d(X, Y) - d| \leq \epsilon$.

Proposition 1: Let $\epsilon < \epsilon'$. If (A, B) is an ϵ-regular pair, then (A, B) is an ϵ'-regular pair.

Proof: All we have done is relax the conditions. ■

Example: Let (A, B) be a random bipartite graph with edge probability p.

Fix $0 < \epsilon \ll p$. Then

$$\begin{aligned}
\Pr\left(\exists a \in A : |\deg(a) - |B|p| \geq \epsilon|B|\right) &\leq |A|\Pr(|\deg(a) - |B|p| \geq \epsilon|B|) \\
&= |A|\Pr\left(\left|\deg(a) - |B|p\right| \geq \frac{\epsilon|B|}{\sqrt{|B|p(1-p)}}\sqrt{|B|p(1-p)}\right) \\
&\leq \exp\left(-\frac{1}{4}\left(\frac{\epsilon|B|}{\sqrt{|B|p(1-p)}}\right)^2\right)
\end{aligned}$$

And as our number of vertices in the graph $n \to \infty$ our last term goes to 0.

Proposition 2: Let (A, B) be an ϵ-regular pair with density d. Let A' be the set of vertices with $\deg(a) \in [(d - \epsilon)|B|, (d + \epsilon)|B|]$. Then $|A'| \geq (1 - 2\epsilon)|A|$.

Proof: Let $X = \{a \in A : \deg(a) < (d - \epsilon)|B|\}$, $Y = B$, $e(X, Y) < (d - \epsilon)|Y||X|$, $d(X, Y) - d < -\epsilon$. Then either $|X| < \epsilon|A|$ or $|Y| < \epsilon|B|$. So $|X| < \epsilon|A|$.

Similarly $|\{a \in A : \deg(a) > (d + \epsilon)|B|\}| < \epsilon|A|$ so $|A'| \geq |A| - 2\epsilon|A|$. ■

Proposition 3: (Intersection property) Let (A, B) be ϵ-regular with density d and $0 < \epsilon \leq d < 1$. If $Y \subseteq B$ and $(d - \epsilon)^{k-1}|Y| \geq \epsilon|B|$ for some $k \geq 1$ then
$$\left|\{(x_1, x_2, ..., x_k) : x_i \in A, \left|Y \cap \left(\bigcap_{i=1}^{k} N(x_i)\right)\right| < (d - \epsilon)^k|Y|\}\right| \leq k\epsilon|A|^k.$$

Proof: By induction on k.

Base case. Look at $k = 1$. $X = \{x_1 \in A : |y \cap N(x_1)| \leq (d - \epsilon)|Y|\}$. $|X| < \epsilon A$ otherwise

$|X| \geq \epsilon|A|, |Y| \geq \epsilon|B|$ and $d(X,Y) - d < -\epsilon$.

Inductive Hypothesis. Assume true for $k-1 (k \geq 2)$.

Let (A,B) be an ϵ-regular pair with density d and $Y \subseteq B, (d-\epsilon)^{k-1}|Y| \geq \epsilon|B|$.

Let $S_1 = \left\{(x_1, x_2, ..., x_{k-1}, x_k) : x_i \in A, \left|Y \bigcap \left(\bigcap_{i=1}^{k-1} N(x_i)\right)\right| \leq (d-\epsilon)^{k-1}|Y|\right\}$.

Certainly $(d-\epsilon)^{k-2}|Y| \geq \epsilon|B|$. By the inductive hypothesis $|S_1| \leq (k-1)\epsilon|A|^{k-1}|A|$.

Let $S_2 =$
$$\left\{(x_1, x_2, ..., x_{k-1}, x_k) : x_i \in A, \left|Y \cap \left(\bigcap_{i=1}^{k} N(x_i)\right)\right| < (d-\epsilon)^k|Y|, \left|Y \cap \left(\bigcap_{i=1}^{k-1} N(x_i)\right)\right| \geq (d-\epsilon)^{k-1}|Y|\right\}$$

Let $Y' = Y \cap \left(\bigcap_{i=1}^{k-1} N(x_i)\right)$.

Then $|Y'| \geq (d-\epsilon)^{k-1}|Y| \geq \epsilon|B|$ (due to the hypothesis assumption).

By the base case $|\{x_k \in A : |Y' \cap N(x_k)| < (d-\epsilon)|Y'|\}| < \epsilon|A|$.

Because $(d-\epsilon)^k|Y| \leq (d-\epsilon)|Y'|$ this set contains $\{x_k \in A : |Y \cap N(x_k) < (d-\epsilon)^k|Y|\}$.

So $|S_2| \leq |A|^k$ and $|S_1| + |S_2| \leq k\epsilon|A|^k$. ■

PROPOSITION 4: Let $A^l(1-k\epsilon) > 1, (d-\epsilon)^k|B| \geq l, (d-\epsilon)^{k-1} \geq \epsilon$. Then any ϵ-regular pair with density d contains a $K_{k,l}$.

PROOF: $\left|\left\{(x_1, x_2, ..., x_k) \in A^k : |\bigcap_{i=1}^{n} N(x_i)\right| \geq (d-\epsilon)^k|B|\right\}\right| > (1-k\epsilon)|A|^k > 1.$

So $\exists (x_1, x_2, ..., x_k)$ such that $\left|\bigcap_{i=1}^{k} N(x_i)\right| \geq (d-\epsilon)k|B| \geq l$ (as long as $(d-\epsilon)^{k-1}|B| \geq \epsilon|B|$).

■

HOMEWORK: (Counts as 2) Prove a general version of the intersection property. Find $f = f(d, \epsilon, k, l)$. Let k, l be integers with the property that $\forall Y \subset B$ if $|Y| \geq f|B|$ then
$$\left|\left\{(x_1, x_2, ..., x_k) \in A^k : \left|Y \cap \left(\bigcap_{i=1}^{l} N(x_i) \cap \bigcap_{i=l+1}^{k} \overline{N(x_i)}\right)\right| < (d-\epsilon)^l(1-d-\epsilon)^{k-l}|Y|\right\}\right| \leq k\epsilon|A|^k.$$

$\overline{N(x_i)} = B \backslash N(x_i)$.

HOMEWORK: Prove I.P. with “ $< (d-\epsilon)^k|Y|$” replaced by “ $> (d+\epsilon)^k|Y|$”. You will need a $|Y| \geq f(d, \epsilon, k)|B|$ condition.

EXAMPLE: Let H be a bipartite graph on $k \times l$ vertices. Then for $0 < \epsilon < d < 1$ if $|A| = |B| = N$ is large enough, an ϵ-regular pair (A, B) contains H as an induced subgraph.