690I Scribe Notes for Jan 26

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February 8, 2006

1 scriven

Clique number, $\omega(G)$. For $\epsilon > 0, b = \frac{1}{p}$, let

$$\hat{k} \pm \epsilon = \lfloor 2\log_b - 2\log_b \log_b(n(1-p)) + 2\log_b(\frac{e}{2}) + 1 \pm \frac{\epsilon}{p} \rfloor$$

Then for $p = p(n)$ such that $p > n^{\delta} \forall_{\delta > 0}$ but $p \leq c$ for some $c < 1$ then

$$k_{-\epsilon} \leq \omega(G_{n,p}) \leq \hat{k} + \epsilon$$

with $prob \to \infty$. Moreover there exists a sequence $\hat{k}(n)$ such that

$$\hat{k}(n) \leq \alpha(G_{n,p}) \leq k(n) + 1, prob \to 1, n \to \infty$$

$$Pr(\alpha(G_{n,p}) = \hat{k}(n) \vee \{\alpha(G_{n,p}) = \hat{k}(n) + 1\}) \to 1, n \to \infty$$

$n \leq \chi(G_{n,p})\alpha(G_{n,p})$
$\frac{n}{\alpha(G)} \leq \chi(G)$ for all graphs.
$\omega(G) \leq \chi(G)$ for all graphs.

$$\alpha(G_{n,p}) \approx \frac{2\log n}{\log(\frac{1}{(1-p)})}$$

Theorem 1 *Let* $p \in 0, 1$

$$\chi(G_{n,p}) \geq (1 - o(1))\frac{n\log(\frac{1}{(1-p)})}{2\log n}$$

Theorem 2 (Bollabas '88) *Let* $p \in (0, 1)$ *be a constant,* $b = \frac{1}{(1-p)}$ *then with high probability,*

$$\frac{n}{2\log_b(n) - \log_b \log_b(n)} \leq \chi(G_{n,p}) \leq \frac{n}{2\log_b(n) - 8\log_b \log_b(n)}$$

Theorem 3 (McDarmond '89) *Let* $p \in (0,1)$ *be a constant,* $b = \frac{1}{(1-p)}$ *then with high probability*

$$\chi(G_{n,p}) = \frac{n}{2\log_b(n) - 2\log_b\log_b(n) + O(1)}$$

Theorem 4 (Luczak '91) $\forall_{p\in p(n)}\exists_{h(n)}$ *such that*
(1) If $p \geq n^{\frac{-6}{7}}, \chi(G_{n,p}) = (1+o(1)h(n)$ *w.h.p.*
(2) If $p < n^{\frac{-6}{7}}, \chi(G_{n,p}) \in \{h(n), h(n)+1\}$ *w.h.p.*

Concentration of χ
Let $\sigma^2 =$ Var(x), μ=E[x].

Theorem 5 *Let* $x \geq 0$

$$Pr(x=0) \leq \frac{\sigma^2}{\mu^2}$$

Theorem 6

$$Pr(x=0) \leq \frac{\sigma^2}{\mu^2+\sigma^2} = \frac{\sigma^2}{E[x^2]}$$

Proof: $(E[x])^2 = [\sum_i iPr(x=i)]^2 \leq (\sum_{i\neq 0} i^2 Pr(x=i))(\sum_{i\neq o} Pr(x=i))$

Theorem 7 (Cauchy-Schwartz) $| < u, v > |^2 \leq ||u||^2||v||^2$
$u =< i\sqrt{Pr(x=i)} >, i \neq 0$
$V =< \sqrt{Pr(x=i)} >, i \neq 0$

$$||u|| = \sqrt{\sum_{i\neq o}(iPr(x=i))^2}$$

$$||v|| = \sqrt{\sum_{\neq o}(iPr(x=i))^2}$$

$$E[x]^2 \leq (\sum_i i^2 Pr(x=i))[1 - Pr(x=0)]$$

$$\frac{E[x]^2}{E[x^2]} \leq 1 - Pr(x=0)$$

QED

Definition 1 (Inedependent random variables) *X and Y are independant if $\forall_{S\subseteq\Omega(x)}$ and $\forall_{T\subseteq\Omega(y)}$*
$Pr(\{x \in S\} \vee \{Y \in T\} = Pr(x \in S)Pr(y \in T)$

Let X and Y be independant random variables then
$E[xy] = E[x] \cdot E[y]$
$Var(x + y) = Var[x] + Var(y)$

Recall $E[x + y] = E[x] + E[y]$ no matter if they are independant or not.

Why independence is needed.
Let $x = \{0, 1\}$ with prob $\frac{1}{2}$ of each.
Thus we have a Bernoulli random variable with probabiliy $\frac{1}{2}$
In general $x = Bern(p)$ is a coin flip with $Pr(x = 1) = p$,$Pr(x = 0) = 1-p$.
$Y = 1 - X$
$E[x] = \frac{1}{2}$
$E[y] = \frac{1}{2}$
$E[xy] = 0$
Because $xy = x(1 - x)$

Check $x = Bern(p)$ then $E[x] = p$, $Var(x) = p(1 - p)$

Proof of discrete case: $E[x]E[y] = (\sum_i iPr(x = i))(\sum_j jPr(y = j))$
$= \sum_i \sum_j ijPr(x = i)Pr(y = j)$
$= \sum_i \sum_j ijPr(\{x = i\} \wedge \{y = j\})$
$= E[xy]$ QED
$Var(x + y) = E[(x + y)^2] - (E[x + y])^2$
$= E[x^2 + 2xy + y^2] - (E[x] + E[y])^2$
$= E[x^2] + 2E[xy] + E[y^2] - (E[x])^2 - 2E[x]E[y]$
$= Var(x) + Var(y) + 2(E[xy] - E[x]E[y])$
$= Var(x) + Var(y)$ by independence QED

The Chernoff Bound
Let $x_1, ..., x_n$ be discrete mutally independant random variables such that $E[x_i] = 0$ and $|x_i| \leq 1$ for all i, $X = \sum_{i=1}^{n} x_i$. Also, let $\sigma_i^2 = Var(x_i)$ and $\sigma^2 = Var(X)$.Then $Pr(x \geq \lambda\sigma) \leq 2e^{\frac{-lambda^2}{4}}$

Observe:
♢ Whoa! Fast dropoff.
♢ $\Sigma^2 = Var[x] = \sum_{i=1}^{n} \sigma_i^2$
♢ mutual independence.

References