We recall that $\alpha(G_{n,p})$ is the size of the largest independent set of vertices in $G_{n,p}$. Now, note that $G_{n,p}$ is a random variable, where two vertices are adjacent to one another with probablility p. We conclude that $\overline{G_{n,p}} = G_{n,1-p}$. Hence, we invoke the previous theorem to note that $\Pr\left[\alpha(G_{n,p}) > \frac{2\ln(n)}{\ln(\frac{1}{1-p})}\right] \to 0$, as $n \to \infty$.

Recall: If $Z \geq 0$, then for $a > 0$, $\Pr(Z \geq a) \leq \frac{\mathrm{E}[Z]}{a}$. (Some say that μ has to be finite here, but it is not necessary because the statement would be vacuously true if it were not.)

Chebyshev's Inequality

Let X be a random variable, and $\mu = \mathrm{E}\left[|X|\right] < \infty$. Then $\Pr(|X-\mu| \geq c) \leq \frac{\mathrm{Var}[X]}{c^2}$.

Proof: Recall that $\mathrm{Var}[X] = \mathrm{E}\left[(X-\mu)^2\right]$. We have that $\Pr(|X-\mu| \geq c) = \Pr\left((X-\mu)^2 \geq c^2\right)$. Applying the Markov inequality, we have that
$\Pr\left((X-\mu)^2 \geq c^2\right) \leq \frac{\mathrm{E}\left[(X-\mu)^2\right]}{c^2}$, as desired. ■

Second Moment Method

Recall that $\mathrm{E}[X]$ is the first moment, $\mathrm{E}\left[X^2\right]$ is the second moment, etc.

But what happens to Chebyshev's Inequality when $c = \mu$? Let $X \geq 0$. Then,

$$\begin{aligned}
\Pr\left(|X-\mu| \geq \mu\right) &= \Pr\left(\{X-\mu \leq -\mu\} \vee \{X-\mu \geq \mu\}\right) \\
&= \Pr\left(\{X \leq 0\} \vee \{X \geq 2\mu\}\right) \\
&\geq \Pr\left(\{X \geq 0\}\right) \\
&= \Pr\left(\{X = 0\}\right)
\end{aligned}$$

Hence,

$$\Pr(X=0) \leq \Pr(|X-\mu| \geq \mu) \leq \frac{\mathrm{Var}(X)}{\mu^2} = \frac{\sigma^2}{\mu^2}$$

Theorem: Fix $p \in (0,1)$. Let $r \sim \frac{2\ln(n)}{\ln(1/p)}(*)$, with $\binom{n}{r} p^{\binom{r}{2}} \to \infty$. Then

$$\Pr\left(\omega(G_{n,p}) \geq r\right) \to 1 \text{ as } n \to \infty$$

Note that to verify $(*)$, we must show that $\forall\, \varepsilon > 0,\ r = \frac{2\ln(n)}{\ln(1/p)}(1-\varepsilon)$ gives $\binom{n}{r} p^{\binom{r}{2}}$.
(Recall that $f \sim g$ where f and g are functions of n means that $\lim_{n\to\infty} \frac{f}{g} = 1$.)

This implies that $\Pr\left[\omega(G_{n,p}) \in \frac{2\ln(n)}{\ln(1/p)}\left(1 \pm o(1)\right)\right] \to 1$ as $n \to \infty$.

We sometimes write this as $\omega(G_{n,p}) \to \frac{2\ln(n)}{\ln(1/p)}$. We say "strong convergence," "with high probability (w.h.p.)," or "almost surely (a.s.)." Any one of these phrases will do, but it is important to note that this only implies a *range* of values.

We are ready to begin the proof.

PROOF: Define $Y_r := |\{S \subseteq V : S \text{ is a clique and } |S| = r\}|$ Then

$$\begin{aligned} \mathrm{E}[Y_r] &= \mathrm{E}\left[\sum_{S\subseteq V} \mathbf{1}_{\{S \text{ clique}\}}\right] \\ &= \sum \mathrm{E}\left[\mathbf{1}_{\{S \text{ clique}\}}\right] \\ &= \sum p^{\binom{r}{2}} \\ &= \binom{n}{r} p^{\binom{r}{2}} \end{aligned}$$

So, by linearity of expectation,

$$\begin{aligned} \mathrm{E}[Y_r^2] &= \mathrm{E}\left[\left(\sum_{S\subseteq V} \mathbf{1}_{\{S \text{ clique}\}}\right)^2\right] \\ &= \mathrm{E}\left[\sum_{S:|S|=r}\sum_{T:|T|=r} \mathbf{1}_{\{S \text{ clique}\}}\mathbf{1}_{\{T \text{ clique}\}}\right] \\ &= \sum_S\sum_T \mathrm{E}\left[\mathbf{1}_{\{S \text{ clique}\}}\mathbf{1}_{\{T \text{ clique}\}}\right] \\ &= \sum_S\sum_T \Pr\left(\{S \text{ clique}\} \wedge \{T \text{ clique}\}\right) \\ &= \sum_{\ell=0}^{r} \binom{n}{r}\binom{r}{\ell}\binom{n-r}{r-\ell} p^{\binom{r}{2}+\binom{r}{2}-\binom{\ell}{2}} \end{aligned}$$

where ℓ is the size of the intersection of the events.

Next,

$$[\mathrm{E}(Y_r)]^2 = \binom{n}{r}^2 p^{2\binom{r}{2}} = \binom{n}{r} p^{2\binom{r}{2}} \sum_{\ell=0}^{r} \binom{r}{\ell}\binom{n-r}{r-\ell}$$

So now we can compute the variance of Y_r.

$$\mathrm{Var}(Y_r) = \mathrm{E}\left[Y_r^2\right] - (\mathrm{E}[Y_r])^2 = \sum_{\ell=2}^{r} \binom{n}{r}\binom{r}{\ell}\binom{n-r}{r-\ell} p^{2\binom{r}{2}} \left[p^{-\binom{\ell}{2}} - 1\right] \ (*)$$

Then,

$$\frac{\mathrm{Var}(Y_r)}{(\mathrm{E}[Y_r])^2} \cdot \frac{*}{\binom{n}{r}^2 p^{2\binom{r}{2}}} = \sum_{\ell=2}^{r} \frac{\binom{r}{\ell}\binom{n-r}{r-\ell}}{\binom{n}{r}} \left[p^{-\binom{\ell}{2}} - 1\right] \ (**)$$

From here, we have a few nightmarish calculations to make!

Notice that if $(**) \to 0$ as $n \to \infty$, then $\Pr(Y_r = 0) \leq \dfrac{\mathrm{Var}(Y_r)}{(\mathrm{E}[Y_r])^2}$, which means that we will have a clique of the desired size! Thus, it only remains to show that $(**) \to 0$.

In $(**)$, label each individual term as $F(\ell)$, so that $(**) = \sum_{l=2}^{r} F(\ell)$.

Observation. $F(\ell) < F(3) + F(r-1)$ for $3 \leq \ell \leq r-1$. (Proof of this can be found in **Random Graphs** by Bollobás.)

Thus,

$$\begin{aligned}
(**) &\leq F(2) + F(r) + (r-3)\left(F(3) + F(r-1)\right) \\
&= \frac{r^4}{2n^2}(p^{-1}-1) + \frac{p^{-\binom{r}{2}-1}}{\binom{n}{r}} + (r-3)\left[\frac{r^6}{6n^3}(p^{-3}-1) + \frac{rnp^{r-1}}{\binom{n}{r}p^{\binom{r}{2}-1}}\right] \\
&\leq \frac{r^4(1-p)}{2n^2p} + \frac{1}{\mu} + r\left[\frac{r^6}{6n^3}(p^{-3}-1) + \frac{rnp^{r-1}}{\binom{n}{r}p^{\binom{r}{2}-1}}\right] \\
&\leq \frac{r^4}{n^2p}\left(\frac{1-p}{2} + \frac{r^3(1-p^3)}{3np^3}\right) + \frac{2}{\mu}\left(\frac{1}{2} + r^2np^r\right)
\end{aligned}$$

Note that $\left(\dfrac{1-p}{2} + \dfrac{r^3(1-p^3)}{3np^3}\right) \leq 1$ if n is large enough. Recall now that $r > -2\log_p n(1-\varepsilon)$ for any $\varepsilon > 0$, so,

$$p^r < p^{-2\log_p n(1-\varepsilon)} = n^{-2+2\varepsilon}$$

Thus,

$$r^2np^r \leq r^2n\left(n^{-2+2\varepsilon}\right) = \frac{r^2}{n^{1-2\varepsilon}} < \frac{1}{2},$$

so long as n is large. Therefore,

$$(**) \leq \frac{r^4}{n^2p} + \frac{2}{\mu}$$

for large n, which implies that $(**) \to 0$ as $n \to \infty$. ■

Next, we have the following theorem, due to Bollobás and Erdős in 1976, which Matula also proved (independently) in that same year.

Theorem: For $\varepsilon > 0$ and $b = 1/p$, set

$$\hat{k}_{\pm\varepsilon} = \left\lfloor 2\log_b n - 2\log_b \log_b\left(n(1-p)\right) + 2\log_b\left(\frac{e}{2}\right) + 1 \pm \frac{\varepsilon}{p}\right\rfloor$$

Then for $p = p(n)$ such that $p > n^{-\delta}\ \forall\ \delta > 0$ but $p \leq c$ for some $c < 1$,

$$\hat{k}_{-\varepsilon} \leq \omega(G_{n,p}) \leq \hat{k}_{+\varepsilon}$$

with probability approaching 1 as $n \to \infty$.

Moreover, there is a sequence $\hat{k}(n)$ such that $\hat{k}(n) \leq \omega(G_{n,p}) \leq \hat{k}(n) + 1$ with high probability.