

Lecture Notes Spring 2006 M690I: Extremal Graph Theory

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1 Bureaucracy

- Introductions all around.
- Statement of syllabus.
- Introduction to the Object of Participation.
- Setting of office hours.
- Thursday: Guest lecturer: M. Axenovich.

2 Terminology

An **extremal** problem is one that asks:

Let \mathcal{G} be a class of graphs. What is the largest (smallest) that parameter P can be over all $G \in \mathcal{G}$

3 Notation

Let $f = f(n)$ and $g = g(n)$. We say that

- $f = O(g)$ if $\exists C$ such that $f(n) \leq C \cdot g(n), \forall n$.
- $f = o(g)$ if $\lim_{n \rightarrow \infty} f/g = 0$.
- $f = \Omega(g)$ if $g = O(f)$. (I.e., $\exists c$ such that $f(n) \geq c \cdot g(n), \forall n$.)
- $f = \omega(g)$ if $g = o(f)$. (I.e., $\lim_{n \rightarrow \infty} f/g = \infty$.)
- $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. (I.e., $\exists c, C$ such that $c \cdot g(n) \leq f(n) \leq C \cdot g(n)$.)
- $f \sim g$ if $\lim_{n \rightarrow \infty} f/g = 1$.

$$\begin{aligned}
1 - x &\leq \exp\{-x\} \\
1 - x &\geq \exp\{-x + x^2\}, & \text{if } x \leq 1/2. \\
\exp\{-x\} &\leq 1 - x + x^2/2, & \text{if } x \geq 0. \\
\exp\{-x\} &\leq 1 - x + 3x^2/4, & \text{if } x \geq -1.
\end{aligned}$$

Theorem 1 (Stirling's formula)

$$\sqrt{2\pi s} \left(\frac{s}{e}\right)^s \leq s! \leq e^{1/12s} \sqrt{2\pi s} \left(\frac{s}{e}\right)^s$$

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \binom{n^k}{k!} \leq \frac{1}{2\sqrt{k}} \left(\frac{en}{k}\right)^k \leq \left(\frac{en}{k}\right)^k$$

Theorem 2 (Boole's inequality) Let S_1, \dots, S_k be a sequence of events in a probability space, then

$$\Pr\left(\bigvee_{i=1}^k S_i\right) \leq \sum_{i=1}^k \Pr(S_i).$$

4 Turán's theorem

Theorem 3 (Mantel, 1907) Let G be a graph with n vertices and no copy of K_3 . Then

$$|E(G)| \leq \left\lfloor \frac{n^2}{4} \right\rfloor \leq \frac{n^2}{4}.$$

Moreover, equality is achieved if and only if G is a bipartite graph with one part of size $\lfloor n/2 \rfloor$ and the other of size $\lceil n/2 \rceil$.

Theorem 4 (Turán, 1941) Let $r \in \mathbb{N}$, $r \geq 2$ and G be a graph with n vertices and no copy of K_{r+1} . Then

$$|E(G)| \leq t(n, r) \leq \left(1 - \frac{1}{r}\right) \frac{n^2}{2},$$

where $t(n, r)$ is the number of edges in $T(n, r)$, the so-called **Turán graph** with the appropriate parameters. The graph $T(n, r)$ is the unique (up to isomorphism) n -vertex r -partite graph such that each part has size either $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Moreover, equality is achieved if and only if $G \cong T(n, r)$.

5 Dirac's theorem

Theorem 5 (Pósa, 1962) Let G be a connected graph with $n \geq 3$ vertices such that for any two nonadjacent vertices x and y we have

$$d(x) + d(y) \geq k.$$

If $k = n$, then G is Hamiltonian, and if $k < n$, then G contains a path of length k and a cycle of length at least $(k + 2)/2$.

Corollary 1 (Originally: Dirac, 1952) *Let G be an n -vertex graph with $\delta(G) \geq n/2$, then G contains a Hamilton cycle.*

6 König-Hall

The following theorem was proven, in an equivalent form

Theorem 6 (König, 1931) *For a graph G , let $\nu(G)$ denote the size of a maximum matching in G and let $\tau(G)$ denote the size of a minimum vertex cover in G . Clearly, $\nu(G) \leq \tau(G)$.*

If G is a bipartite graph, then $\nu(G) = \tau(G)$.

The **Term Rank** of a $(0, 1)$ -matrix is the largest number of 1s that can be chosen from the matrix such that no 2 selected 1s lie on the same line. A set S of rows and columns is a **cover** of a $(0, 1)$ -matrix if the matrix becomes a zero matrix after all the lines in S have been deleted.

Theorem 7 (König-Egerváry 1931) *The term rank of a $(0, 1)$ -matrix is the cardinality of its smallest cover.*

For any graph G , let $N(x)$ be the neighborhood of vertex x and $N(S) = \bigcup_{x \in S} N(x)$. A matching \mathcal{M} is said to **saturate** set W if each vertex in w is incident to some edge in \mathcal{M} .

Theorems 6 and 7 are both equivalent to the following convenient statement:

Theorem 8 (Hall, 1935) *Let G be a bipartite graph with vertex sets V_1 and V_2 . G contains a matching that saturates V_1 if and only if*

$$|N(S)| \geq |S|, \quad \forall S \subseteq V_1.$$

The author is Philip Hall, contradicting what I said in class. Marshall Hall was an unrelated contemporary of Philip.

There are other equivalent forms. For example, Dilworth's theorem with respect to partially-ordered sets (smallest chain cover equals largest antichain) implies Theorem 8. In this class, we will refer to Theorem 6 or Theorem 8 as "König-Hall".

Corollary 2 *Let G be a bipartite graph with vertex sets V_1 and V_2 and $d \in \mathbb{N}$ with*

$$|N(S)| \geq |S| - d, \quad \forall S \subseteq V_1,$$

then G contains a matching of size $|V_1| - d$.

HOMEWORK 1 Let G be a k -partite graph with vertex sets V_1, \dots, V_k , $|V_1| = \dots = |V_k| = N$ and the minimum degree of a vertex in the graph induced by (V_i, V_j) is at least $(1 - 2^{1-k})N$. Then, G has a subgraph that consists of N vertex-disjoint copies of K_k .

Theorem 9 (Tutte, 1947) Let G be a graph and, for any subgraph H , let $q(H)$ denote the number of connected components of H that have odd order. Then, G has a 1-factor if and only if

$$q(G \setminus S) \leq |S|, \quad \forall S \subseteq V(G).$$

7 The Hajnal-Szemerédi theorem

A coloring is called **proper** if no edge has two endpoints the same color. A **spanning subgraph** of graph G is a subgraph H which has $V(H) = V(G)$.

Theorem 10 (Hajnal-Szemerédi, 1970) If $n = s(r + 1)$, G is an n -vertex graph and $\Delta(G) \leq r$, then G has a proper $(r + 1)$ -coloring with equal color classes.

Equivalently,

Corollary 3 Let G be an n -vertex graph. If s divides n and $\delta(G) \geq (1 - \frac{1}{s})n$, then G has a spanning subgraph that consists of vertex-disjoint subgraphs which are each copies of K_s .

The proof is long. It should be noted that both Dirac and Hajnal-Szemerédi imply that $2 \mid n$ and $\delta(G) \geq n/2$ imply that G has a (spanning) 1-factor.

8 Corrections

The “Hall” in Hall’s matching theorem (König-Hall) is Philip Hall. Marshall Hall was an unrelated contemporary of Philip’s.

The homework was not so much incorrect, but we can do better, and I have corrected it. The minimum degree in a pair (V_i, V_j) being only $(1 - 2^{1-k})N$ is sufficient to get the answer.

9 Ramsey’s theorem

Theorem 11 (Ramsey, 1930) *For all $k \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that every graph with at least n vertices with K_k or $\overline{K_k}$ as a(n induced) subgraph.*

The smallest such n is called $R(k, k)$.

It is hard to prove this theorem, it is actually easier to prove a generalization.

Theorem 12 *For all $s \in \mathbb{N}$, and all constants $r_1, \dots, r_s \in \mathbb{N}$, there exists an $n \in \mathbb{N}$ such that every s -coloring of the edges of a complete graph with at least n vertices will have a K_{r_i} in color i for at least one value of $i \in \{1, \dots, s\}$.*

The smallest such n is called $R(r_1, \dots, r_s)$.

Proof of Theorem 12. Clearly, we can permute the arguments of R and arrive at the same answer. It should be obvious that if all $r_i \geq 3$, then

$$R(r_1, \dots, r_s) \leq R(r_1 - 1, r_2, \dots, r_s) + \dots + R(r_1, \dots, r_{s-1}, r_s - 1).$$

Furthermore, $R(2, \dots, 2, r_1, \dots, r_t) \leq R(2, \dots, 2, r_1 - 1, r_2, \dots, r_t) + \dots + R(2, \dots, 2, r_1, \dots, r_{t-1}, r_t - 1)$ and $R(2, r) = r$. \square

Theorem 13 *Let $k, \ell \in \mathbb{N}$, $k, \ell \geq 2$. Then,*

$$R(k, \ell) \leq \binom{k + \ell - 2}{k - 1}.$$

So,

$$R(k, k) \leq \frac{c}{\sqrt{k}} 4^k,$$

for some constant c .

Theorem 14 *If $k \in \mathbb{N}$, $k \geq 2$, then $R(k, k) > n$ if $\binom{n}{k} 2^{1 - \binom{k}{2}} < 1$.*

Thus, $R(k, k) > \frac{k}{e\sqrt{2}} 2^{k/2}$.

Proof. Here, instead of edges and non-edges, we let every pair of vertices have an edge between them and be colored either red or blue. Between each pair of

vertices, determine randomly whether or not an edge is red or blue.

$$\Pr \left(\bigwedge_{\substack{S \subseteq V \\ |S|=k}} \{S \text{ is monochromatic}\} \right) \leq \sum_{\substack{S \subseteq V \\ |S|=k}} \Pr(S \text{ is monochromatic}) \\ \leq \binom{n}{k} 2^{1-\binom{k}{2}}$$

If this probability is less than 1, then $R(k, k) > n$. So, we want the largest n such that

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1.$$

Using the fact that $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$, we get the second result. \square

Spencer (1975) improved the lower bound by a factor of 2.

The following is the asymptotic state of the art:

$$\sqrt{2} \leq \liminf_{k \rightarrow \infty} R(k, k)^{1/k} \leq \limsup_{k \rightarrow \infty} R(k, k)^{1/k} \leq 4$$

It is not known if the limit even exists. One (asymptotically) settled result is:

Theorem 15 (Kim, 1995) *Let $k \geq 3$. Then,*

$$R(3, k) = \Theta(k^2 / \log k).$$

10 Another example of probability

Theorem 16 *For every graph G , there exists a bipartite subgraph of G that contains at least half the edges of G .*

Proof. Partition $V(G)$ into two nonempty pieces V_1 and V_2 . If a vertex $v \in V_1$ is adjacent to more vertices in V_1 than V_2 , let the new partition be $V_1 \setminus \{v\}$ and $V_2 \cup \{v\}$. With each step, this algorithm increases the number of edges between parts. Hence, it terminates. \square

Theorem 17 *For every graph G and every $k \in \mathbb{N}$, there exists a k -partite subgraph of G that contains at least $(1 - k^{-1}) |E(G)|$ edges.*

Trying to generalize the above proof is complicated, there's a better solution.

Proof. Color each vertex uniformly at random from among k colors. The probability that any given edge is bi-colored is $(k-1)/k$. The expected number of bi-colored edges is $\frac{k-1}{k} |E(G)|$. Thus, some coloring must have at least this many bi-colored edges. Let the color classes of this coloring be our k partitions. \square

11 Proofs

11.1 Proof of Turán's theorem

Proof of Theorem 4. We proceed by induction on n . If $n \leq r$, then $t(n, r) = \binom{n}{2}$.

Now, let $n > r$ and let G be maximal with respect to not having a K_{r+1} . If $n > r$, then G must have a copy of K_r , otherwise we could add edges. Let the vertex set of G be V , the vertex set of some K_r be A and $B = V \setminus A$.

We use the fact that no vertex in B can be adjacent to more than $r - 1$ members of A , otherwise we have a K_{r+1} . Hence $|E(G[A, B])| \leq (n - r)(r - 1)$.

$$\begin{aligned} |E(G)| &= |E(G[A])| + |E(G[A, B])| + |E(G[B])| \\ &\leq \binom{r}{2} + (n - r)(r - 1) + t(n - r, r) \\ &= t(n - r, r) + n(r - 1) - \binom{r}{2} \end{aligned}$$

It suffices to show that, whenever $n > r$, $t(n, r) = t(n - r, r) + n(r - 1) - \binom{r}{2}$, an exercise.

Proving the uniqueness of the best possible solution can be done by this inductive step as well. We will spare you. \square

11.2 Proof of Dirac (Pósa)

Note that

HOMEWORK 2 For every $n \in \mathbb{N}$, $n \geq 3$, there is a graph G with $\delta(G) = \lfloor \frac{n-1}{2} \rfloor$ but G is not Hamiltonian.

Proof of Theorem 5. Let P be a maximum-length path, $P = x_0, x_1, \dots, x_\ell$. If there is a cycle with the same vertex set as P , then $\ell = n - 1$ because the connectivity of the graph would give that an edge between a y and $V(P)$ would give a path of length $\ell + 1$. That circumstance would complete the theorem.

So, assume there is no cycle on the vertices of P . The maximality of P establishes that $N(x_0), N(x_\ell) \subseteq V(P)$. Moreover, denote $N^+(x_\ell)$ to be the set of x_{i+1} for which $x_i \sim N(x_\ell)$. Since there is no cycle, $N(x_0) \cap N^+(x_\ell)$, otherwise $x_{i+1} \in N(x_0) \cap N^+(x_\ell)$ implies that $x_0, \dots, x_i, x_\ell, x_{\ell-1}, \dots, x_{i+1}, x_0$ is a cycle. Since $x_0 \notin N(x_0) \cap N^+(x_\ell)$,

$$\ell + 1 \geq 1 + |N(x_0)| + |N^+(x_\ell)| \geq k + 1.$$

As to the cycle, there is one formed by the initial segment up to the last neighbor of x_0 or the final segment starting with the first neighbor of x_ℓ . So, there is a cycle of size at least

$$\max\{|N(x_0)| + 1, |N(x_\ell)| + 1\} \geq \left\lceil \frac{k}{2} \right\rceil + 1 \geq \frac{k + 2}{2}.$$

\square

11.3 Proof of König-Hall

Proof of Theorem 8. One direction is easy: If there exists an $S \subseteq V_1$ for which $|N(S)| < |S|$, then no matching can saturate S .

Now, suppose $|N(S)| \geq |S|$ for all $S \subseteq V_1$. Let \mathcal{M} be a maximum matching in G and $V(\mathcal{M}) \cap V_1 = S$. Define an **alternating path** to be a path in G that alternates between edges in \mathcal{M} and edges outside of \mathcal{M} . Define an **augmenting path** to be one in which the first and last are outside of \mathcal{M} and the endpoints have no edge of \mathcal{M} incident to them.

Clearly, if \mathcal{M} is maximum, then no augmenting path can exist because we could just take that path, P and remove the \mathcal{M} edges in favor of the others. What results is a larger matching.

Let $x \in V_1 \setminus S$. Define T as all of those vertices (in V_2) which are reachable from x via an alternating path. Clearly, $T \subseteq V(\mathcal{M}) \cap V_2$, otherwise there would be an augmenting path. But the vertices in T have a neighbor in \mathcal{M} in S . Call these vertices S' . If any vertex in S' has a neighbor outside of T , then we have an alternating path to that neighbor, a contradiction. \square