Homework 3

Spring 2006 M690I: Extremal Graph Theory

Due Apr. 20, 2006, assigned Apr. 6, 2006
Typo fixed Apr. 12, 2006

Please hand in solutions for 2 of the following problems. Your writeup must be your own, but I cannot object to your discussing problems with others. Please give credit if you work with someone else and do at least one problem that your collaborators do not do.

Every extra problem you do will make a check-minus into a coveted check as long as your solution is in TEX.

If you turn this in by April 13, I will grade and return it to you by April 18. If it is sufficient, then you are done. If not, you can fix the mistakes and turn the final copy in April 25 with no penalty.

HOMEWORK 1 *Let $G = (V_1, V_2, V_3; E)$ be a tripartite graph with $|V_1| = |V_2| = |V_3| = 2N$ and each vertex in V_i is adjacent to at least N vertices in V_j for all distinct i, j in $\{1, 2, 3\}$. Prove that either G has a triangle or G must be a tripartite graph that is unique. Describe it.*

HOMEWORK 2 *Turán's theorem states that the graph that has the most number of edges from among all K_{k+1}-free graphs on n vertices is $T(n,k)$, a k-partite graph with all sets of size as equal as possible. If $t(n,k) = |E(T(n,k))|$ prove that*

$$t(n,k) = \frac{n^2}{2}\left(\frac{k-1}{k}\right) - \frac{k}{2}\left(\left\lceil\frac{n}{k}\right\rceil - \frac{n}{k}\right)\left(\frac{n}{k} - \left\lfloor\frac{n}{k}\right\rfloor\right).$$

Verify that

$$\frac{n^2}{2}\left(\frac{k-1}{k}\right) - \frac{k}{8} \leq t(n,k) \leq \frac{n^2}{2}\left(\frac{k-1}{k}\right)$$

and improve the lower bound if k is odd.

HOMEWORK 3 *Prove the Slicing Lemma:*

Lemma 1 (Slicing Lemma) *Let us be given ϵ, α, d such that $\epsilon > 0$, $1 > \alpha > \epsilon$ and $d, 1-d \leq \max\{2\epsilon, \epsilon/\alpha\}$. Let $A' \subseteq A$ with $|A'| \geq \alpha|A|$ and $B' \subseteq B$ with $|B'| \geq \alpha|B|$. Then (A', B') is ϵ'-regular with $\epsilon' = \max\left\{2\epsilon, \frac{\epsilon}{\alpha}\right\}$ and density in $(d-\epsilon, d+\epsilon)$.*

HOMEWORK 4 *Prove the following proposition:*

Proposition 1 *Let $\epsilon > 0$ and $\epsilon'' > \epsilon$. Let (A, B) be an ϵ-regular pair with density d and $|A| = |B| = L$. Chose X uniformly at random from $\binom{A}{L_0}$ and Y uniformly from $\binom{B}{L_0}$, where $L_0 > \epsilon'' L$. With high probability (i.e., probability approaching 1 as $L \to \infty$), each of the following holds:*

1. $|\{y \in Y : |X \cap N(y)| \leq (d - \epsilon'')|X|\}| \leq \epsilon''|Y|$
2. $|\{y \in Y : |X \cap N(y)| \geq (d + \epsilon'')|X|\}| \leq \epsilon''|Y|$
3. $\left|\left\{(y_1, y_2) \in Y \times Y : |X \cap N(y_1) \cap N(y_2)| \leq (d - \epsilon'')^2|X|\right\}\right| \leq 2\epsilon''|Y|^2$
4. $\left|\left\{(y_1, y_2) \in Y \times Y : |X \cap N(y_1) \cap N(y_2)| \geq (d + \epsilon'')^2|X|\right\}\right| \leq 2\epsilon''|Y|^2$

These four statements also hold if the roles of X and Y are switched.

You only need to prove the 4 items. Switching X and Y results from a perfectly symmetric argument.

HOMEWORK 5 *Let G be a d-regular graph on n vertices with spectrum $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$. Prove each of the following. You can use any theorems or statements given in class.*

- $\lambda_0 = d$
- $\lambda_0 > \lambda_1$ *if and only if G is connected.*
- $\lambda_0 = -\lambda_{n-1}$ *if and only if G is bipartite.*

HOMEWORK 6 *Prove that if G is a d-regular graph on n vertices with spectrum $d = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$, then the complement $\overline{G}$ is a $n - d - 1$-regular graph on n vertices with spectrum*

$$n - d - 1 \geq -\lambda_{n-1} - 1 \geq \cdots \geq -\lambda_1 - 1$$