Homework 2

Spring 2006 M690I: Extremal Graph Theory

Due Mar. 23, 2006, assigned Mar. 09, 2006 (modified Mar. 20, Mar. 22)

Please hand in solutions for 3 of the following problems. Your writeup must be your own, but I cannot object to your discussing problems with others. Please give credit if you work with someone else and do at least one problem that your collaborators do not do.

HOMEWORK 1 *Do both of the following:*

- *Prove that the nonedges in an ϵ-regular pair form an ϵ-regular pair. What is the density?*
- *Prove the following:*

  Theorem 1 *Let $\epsilon < 1/2$ and $d > 2\epsilon$. Let (A, B) be an ϵ-regular pair of density d with $|A| = |B| = L$. Then, there exist sets $A' \subseteq A$ and $B' \subseteq B$ such that $|A'|, |B'| \geq (1-\epsilon)L$ such that (A', B') is $(\epsilon, d-2\epsilon)$-super-regular with density in $(d-\epsilon, d+\epsilon)$.*

HOMEWORK 2 *Find sufficient conditions on δ, ϵ, $|Y|$ and $|B|$ so as to form the other direction of the intersection property. That is,*

$$\left|\left\{(a_1, \ldots, a_k) \in A^\ell : \left|Y \cap \bigcap_{i=1}^{k} N(a_i)\right| \geq (d+\epsilon)^k |Y|\right\}\right| \leq k|A|^k.$$

HOMEWORK 3 *This replaces #2 as well (it counts as 2 problems): Completely generalize the intersection property. Find sufficient conditions on δ, ϵ, $|Y|$ and $|B|$ so as to form the general intersection property. That is, for any $\ell \in \{0, \ldots, k\}$,*

$$\left|\left\{(a_1, \ldots, a_k) \in A^\ell : \left|Y \cap \bigcap_{i=1}^{\ell} N(a_i) \cap \bigcap_{i=\ell+1}^{k} N(a_i)\right| \leq (d-\epsilon)^\ell (1-d-\epsilon)^{k-\ell} |Y|\right\}\right| \leq k|A|^k$$

$$\left|\left\{(a_1, \ldots, a_k) \in A^\ell : \left|Y \cap \bigcap_{i=1}^{\ell} N(a_i) \cap \bigcap_{i=\ell+1}^{k} N(a_i)\right| \geq (d+\epsilon)^\ell (1-d+\epsilon)^{k-\ell} |Y|\right\}\right| \leq k|A|^k$$

For completeness:

Theorem 2 (RegLemDF) *For all $\epsilon > 0$, there exists an $M = M(\epsilon)$ such that if $G = (V, E)$ is a graph on n vertices and $d \in [0, 1]$ is any real number then there exists a partition of the vertex set $V = V_0 + V_1 + \cdots + V_\ell$ and there exists a subgraph G' of G with*

- $\ell \leq M$
- $|V_0| \leq \epsilon n$
- *all clusters V_i, $i \geq 1$ are of the same size $L \leq \epsilon n$*
- $e(G'[V_i]) = 0$, $\forall i \geq 1$
- $\deg_{G'}(v) > \deg_G(v) - (d + \epsilon)n$
- ***all** pairs (V_i, V_j), $1 \leq i < j \leq \ell$ are ϵ-regular with density either 0 or at least d.*

HOMEWORK 4 *Prove the degree form of the Regularity Lemma from the original statement.* **Hints:**

1. *Apply RegLem with $(\epsilon')^2$ and $m = \lceil (\epsilon')^{-1} \rceil$.*
2. *Remove all edges in a cluster.*
3. *Place all clusters in at least $\epsilon'\ell$ irregular pairs into the leftover set.*
4. *Remove all edges in the remaining irregular pairs.*
5. *Place into the leftover set all vertices with $\deg_{U_i}(v) \geq (d + \epsilon')n$ where*

   $$U_i = \bigcup_{j \in \{1, \ldots, \ell\}} \{V_j : d(V_i, V_j) \leq d\}$$

   and some more vertices to make the remaining clusters of the same size.
6. *Remove all edges in sparse pairs.*
7. *Check $\deg_{G'}(v) > \deg_G(v) - (d + 3\epsilon')n$ and the new $|V_0| \leq 3\epsilon' n$.*

HOMEWORK 5 *Prove the following:*

Theorem 3 *Let H be a graph with h vertices and $\chi(H) = p$. Let $\beta > 0$ and $\epsilon = (\beta/6)^h$. If n is large enough and a graph on n vertices, G_n has*

$$e(G_n) \geq \left(1 - \frac{1}{p-1} + \beta\right) \frac{n^2}{2}$$

then

$$\|H \to G_n\| \geq \left(\frac{\epsilon n}{M(\epsilon)}\right)^h,$$

where $M(\epsilon)$ is the constant from RegLemDF.

Hint: *Use the Key Lemma from class. Mimic the in-class proof of Erdős-Stone. I.e., find a K_p in the reduced graph. Apply the Key Lemma with t being the maximum size of a color class in a p-coloring of the vertices of H. (So, we can guarantee that $t \leq h - p + 1$.)*

HOMEWORK 6 *Let G be a graph with vertex set $\binom{[2k+1]}{k}$ for some integer k such that two vertices are adjacent if and only if their corresponding sets are* ***not*** *disjoint. Prove or disprove that if $k \geq 2$, then this graph is diameter-2 critical. (Note, you have to prove that it both has diameter 2 and that it is critical with that property with respect to edge-deletion.)*

What kind of graph occurs when $k = 1$? Is that diameter-2 critical? Why or why not?

Note, number 1, number 4 has a revision of U_i and number 6 have changed from the previous version. In 1, $\epsilon < 1/2$ and in 6 the sets should be nondisjoint and it is now a prove or disprove question.