Homework 1

Spring 2006 M690I: Extremal Graph Theory

Due Feb. 9, 2006, assigned Jan. 26, 2006

Please hand in solutions for 3 of the following problems. Your writeup must be your own, but I cannot object to your discussing problems with others. Please give credit if you work with someone else and do at least one problem that your collaborators do not do.

HOMEWORK 1 *Let G be a k-partite graph with vertex sets $V_1, \ldots, V_k$, $|V_1| = \cdots = |V_k| = N$ and the minimum degree of a vertex in the graph induced by (V_i, V_j) is at least $\left(1 - 2^{1-k}\right) N$. Prove that G has a subgraph that consists of N vertex-disjoint copies of K_k.*

HOMEWORK 2 *For every $n \in \mathbb{N}$, $n \geq 3$, show that there is a graph G with $\delta(G) = \left\lfloor \frac{n-1}{2} \right\rfloor$ but G is not Hamiltonian.*

HOMEWORK 3 **Prove or disprove.** *For every positive integer n and every graph G of order n with $e(G)$ vertices, there is a bipartite subgraph of G with at least*

$$e(G) \frac{\left\lfloor \frac{n^2}{4} \right\rfloor}{\binom{n}{2}}$$

edges.

HOMEWORK 4 *Prove linearity of expectation. That is, let $X_1, \ldots, X_n$ be random variables with $\max_i \{\mathbb{E}[|X_i|]\} < \infty$ and $a_1, \ldots, a_n$ be real numbers. Then*

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i].$$

You only need to prove this for integer-valued random variables, even though it is true in arbitrary measure 1 spaces.

HOMEWORK 5 *Fix $p \in (0, 1)$. Show that for all $\epsilon > 0$,*

- *If $r = \frac{(1+\epsilon) 2 \log n}{-\log p}$, then $\binom{n}{r} p^{\binom{r}{2}} \to 0$.*
- *If $r = \frac{(1-\epsilon) 2 \log n}{-\log p}$, then $\binom{n}{r} p^{\binom{r}{2}} \to \infty$.*

Both limits are taken as $n \to \infty$.