

M608X (S08) Homework 3

Due date: 24 April 2008 (paper copy); 5 May 2008 (electronic copy)

Do at least 3 of the following:

1. Let $G = (V_1, V_2, V_3; E)$ be a tripartite graph with $|V_1| = |V_2| = |V_3| = 2N$ and each vertex in V_i is adjacent to at least N vertices in V_j for all distinct i, j in $\{1, 2, 3\}$. Prove that either G has a triangle or G must be a tripartite graph that is unique. Describe it.
2. Prove the Slicing Lemma:

Lemma 1 (Slicing Lemma) *Let us be given ϵ, α, d such that $\epsilon > 0$, $1 > \alpha > \epsilon$ and $d, 1 - d \leq \max\{2\epsilon, \epsilon/\alpha\}$. Let $A' \subseteq A$ with $|A'| \geq \alpha|A|$ and $B' \subseteq B$ with $|B'| \geq \alpha|B|$. Then (A', B') is ϵ' -regular with $\epsilon' = \max\{2\epsilon, \frac{\epsilon}{\alpha}\}$ and density in $(d - \epsilon, d + \epsilon)$.*

3. Prove the following proposition:

Proposition 0.1 *Let $\epsilon > 0$ and $\epsilon'' > \epsilon$. Let (A, B) be an ϵ -regular pair with density d and $|A| = |B| = L$. Chose X uniformly at random from $\binom{A}{L_0}$ and Y uniformly from $\binom{B}{L_0}$, where $L_0 > \epsilon''L$. With high probability (i.e., probability approaching 1 as $L \rightarrow \infty$), each of the following holds:*

- (a) $|\{y \in Y : |X \cap N(y)| \leq (d - \epsilon'')|X|\}| \leq \epsilon''|Y|$
- (b) $|\{y \in Y : |X \cap N(y)| \geq (d + \epsilon'')|X|\}| \leq \epsilon''|Y|$
- (c) $|\{(y_1, y_2) \in Y \times Y : |X \cap N(y_1) \cap N(y_2)| \leq (d - \epsilon'')^2|X|\}| \leq 2\epsilon''|Y|^2$
- (d) $|\{(y_1, y_2) \in Y \times Y : |X \cap N(y_1) \cap N(y_2)| \geq (d + \epsilon'')^2|X|\}| \leq 2\epsilon''|Y|^2$

These four statements also hold if the roles of X and Y are switched.

You only need to prove the 4 items. Switching X and Y results from a perfectly symmetric argument.

4. **Note that this problem counts as two.**

Recall:

Theorem 0.1 (SzemRegLemDF) *For all $\epsilon > 0$, there exists an $M = M(\epsilon)$ such that if $G = (V, E)$ is a graph on n vertices and $d \in [0, 1]$ is any real number then there exists a partition of the vertex set $V = V_0 + V_1 + \dots + V_\ell$ and there exists a subgraph G' of G with*

- $\ell \leq M$
- $|V_0| \leq \epsilon n$
- all clusters $V_i, i \geq 1$ are of the same size $L \leq \epsilon n$
- $e(G'[V_i]) = 0, \forall i \geq 1$
- $\deg_{G'}(v) > \deg_G(v) - (d + \epsilon)n$
- **all** pairs $(V_i, V_j), 1 \leq i < j \leq \ell$ are ϵ -regular with density either 0 or at least d .

Prove the degree form of the Regularity Lemma from the original statement. **Hints:**

- Apply RegLem with $(\epsilon')^2$ and $m = \lceil (\epsilon')^{-1} \rceil$.
- Remove all edges in a cluster.
- Place all clusters in at least $\epsilon' \ell$ irregular pairs into the leftover set.
- Remove all edges in the remaining irregular pairs.
- Place into the leftover set all vertices with $\deg_{U_i}(v) \geq (d + \epsilon')n$ where

$$U_i = \bigcup_{j \in \{1, \dots, \ell\}} \{V_j : d(V_i, V_j) \leq d\}$$

and some more vertices to make the remaining clusters of the same size.

- Remove all edges in sparse pairs.
- Check $\deg_{G'}(v) > \deg_G(v) - (d + 3\epsilon')n$ and the new $|V_0| \leq 3\epsilon'n$.

Good problems from HW2 (not eligible, just for your own enlightenment):

- Prove that the property that $G \sim G(n, p)$ has no isolated vertices has threshold function $\ln n/n$.