M608X (S08) Homework 2

Due date: 6 March 2008

Do at least 4 of the following:

1. Prove the following:

   - If G is a graph on n vertices with no isolated vertices, then G has a dominating set of size at most $\lfloor n/2 \rfloor$.
   - For any positive integer n and any δ, $1 \leq \delta \leq n-1$, construct a simple graph on n vertices with minimum degree δ and no dominating set smaller than $\left\lfloor \frac{n}{\delta+1} \right\rfloor$.
   - For any positive integer n and any even δ, $1 \leq \delta \leq n-1$, construct a simple graph on n vertices with minimum degree n and no dominating set smaller than $2\left\lfloor \frac{n}{\delta+2} \right\rfloor$.

2. Prove the following:

   - If $X_1, \ldots, X_n$ are *pairwise* independent random variables, each with finite mean, then

   $$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i).$$

   - If $X_1, \ldots, X_n$ are *mutually* independent, discrete random variables, then

   $$\mathbb{E}\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} \mathbb{E}[X_i].$$

3. Prove that, for a random variable X, $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$.

4. Prove the following:

   - If α is a positive real number and y is a real number such that $|y| \leq 1$, then

   $$e^{\alpha y} \leq \cosh(\alpha) + \sinh(\alpha) y. \quad (1)$$

   Hence, if Y is a random variable with $|Y| \leq 1$, then $\mathbb{E}[e^{\alpha Y}] \leq \mathbb{E}[\cosh(\alpha) + \sinh(\alpha) Y] = \cosh(\alpha) + \sinh(\alpha)\mathbb{E}[Y]$.

- If x is a real number, then $\cosh(x) \leq e^{x^2/2}$.

5. Do the following:

   - The minimum rank of an n-vertex graph G, denoted $\mathrm{mr}(G)$, is the minimum rank over all $n \times n$ matrices A such that, for distinct i, j, if $A_{ij} = 0$ then vertex i is nonadjacent to vertex j in G. Note that the diagonal entries in A can be arbitrary. A **principle minor** (of size $n-1$) of A is obtained by deleting the i^{th} row and i^{th} column of A. Prove that, for any principle minor, A' of A that

     $$\mathrm{rank}(A) - 2 \leq \mathrm{rank}(A') \leq \mathrm{rank}(A).$$

     Use this fact to show that, if $G \sim G(n,p)$ then $\mathrm{mr}(G)$ is tightly concentrated around its mean. Find the proper expression for the probability. It has been recently showed by Bryan Shader that $\mathbb{E}\left[G(n,1/2)\right] \geq n/14$.

   - Let $\mathcal{H}$ be an r-uniform hypergraph with m hyperedges. Prove that $\mathcal{H}$ has Property B if $m < 2^{r-1}$.

6. Use the symmetric version of the Lovász Local Lemma to prove that $R(k,k) > \frac{\sqrt{2}}{e}(1+o(1))k2^{k/2}$. This is an improvement of a multiplicative factor of 2 over the naïve probabilistic bound in Erdős' 1947 paper.

7. Do the following:

   - An amusing philosophical diversion is the **St. Petersburg paradox**. The game is simple. A coin is flipped until heads occurs. If heads occurs on the first flip, the house pays \$2. If the first flip is tails and the second heads, the house pays $\$2^2$=\$4. In general, if heads occurs first on the i^{th} flip, then the house pays $\$2^i$. Would you pay \$1000 to play this game? Why or why not?

   - Write (and solve) a "good" homework problem based on weeks 3 through 7.

Good problems from HW1 (not eligible, just for your own enlightenment):

- Suppose that we have $3p$ points in the plane and that, for any two points x and y, we have $\|x-y\|_2 \leq 1$. Show that at most $3p^2$ of the points are separated by a distance greater than $\sqrt{2}/2$.

- Use Ramsey's theorem to show that for any $k, \ell \in \mathbb{N}$, there exists an n such that every sequence of n distinct integers contains an increasing subsequence of length $k+1$ or a decreasing subsequence of length $\ell+1$. (Ex. 9.6 in Diestel)