TAO’S SPECTRAL PROOF OF THE SZEMERÉDI REGULARITY
LEMMA

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ABSTRACT. On December 3, 2012, following the Third Abel conference, in honor of Endre
Szemerédi, Terence Tao posted on his blog a proof of the spectral version of Szemerédi’s
regularity lemma. This, in turn, proves the original version.

1. Introduction

Tao attributes this proof to Frieze and Kannan [1].

One thing to observe is that this is a statement on matrices and graphs are just a conse-
quence.

2. Basic matrix version

Lemma 1 (Szemerédi’s regularity lemma, matrix version). Let $T$ be a self-adjoint $n \times n$
matrix such that $\text{tr}(T^2) \leq n^2$. Let $V$ be the set of $n$ indices and let $\epsilon > 0$. Then there exists
an $M \leq M(\epsilon)$ and

- a decomposition of $T$ into three matrices, $T = T_1 + T_2 + T_3$, each of which is self-
  adjoint,
- a partition $V = V_0 \cup V_1 \cup \cdots \cup V_M$, and
- a set of pairs $\Sigma \subset \binom{\{0,\ldots,M\}}{2}$ (which contains all pairs with 0),

such that

- for all $i, j \in \{0, 1, \ldots, M\}$, there exists $d_{ij}$ such that for all $a \in V_i$ and $b \in V_j$, we
  have $|(T_1)_{ab} - d_{ij}| < \epsilon$,
- for all $i, j \in \{0, 1, \ldots, M\} - \Sigma$, we have $\sum_{a \in V_i} \sum_{b \in V_j} |(T_2)_{ab}|^2 < \epsilon|V_i||V_j|$, and
- for all $i, j \in \{0, 1, \ldots, M\} - \Sigma$, we have $n \cdot \sigma(T_3) < \epsilon|V_i||V_j|$, (where $\sigma(T_3)$ is $T_3$’s
  largest singular value) and
- $\sum_{(i,j) \in \Sigma} |V_i||V_j| \leq \epsilon n^2$.

Proof. Enumerate $V = \{1, \ldots, n\}$. Since $T$ is self-adjoint, it has an eigenvalue decomposition

$$T = \sum_{i=1}^{n} \lambda_i u_i u_i^*,$$

for some orthonormal basis $u_1, \ldots, u_n$ of $\mathbb{C}^n$ (where the vectors are column vectors) and real
eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. We arrange them in decreasing order of magnitude

$$|\lambda_1| \geq \cdots \geq |\lambda_n|.$$
In a self-adjoint matrix, the trace of $T^2$ is the sum of the squares of the eigenvalues of $T$ and so $\text{tr}(T^2) = \sum_{j=1}^{n} |\lambda_j|^2$. So we can bound the eigenvalues by observing that $i\lambda_i^2 \leq \sum_{j=1}^{i} \lambda_j^2 \leq n^2$ and so, for all $i \in \{1, \ldots, n\}$,

$$|\lambda_i| \leq \frac{n}{\sqrt{i}}. \tag{1}$$

We will be given a function $F : \mathbb{N} \to \mathbb{N}$ that we will specify later. This function does depend on $\epsilon$ (we suppress this in the notation) and it satisfies the inequality $F(i) > i$ for all integers $i$. We want to find an integer $J$ for which

$$\sum_{J \leq j < F(J)} |\lambda_j|^2 \leq \epsilon^3 n^2. \tag{2}$$

To do this, we consider the partition of $\{1, \ldots, n\}$ into intervals $[F^{(k-1)}(1), F^{(k)}(1) - 1]$ from $k = 1, \ldots, 1/\epsilon^3$ where $F^{(k)}$ represents the $k$th composition of $F$ with itself. Note that either we find a $J = F^{(k-1)}(1)$ for which (2) is satisfied or the sum of $|\lambda_j|^2$ for all $j$ in some interval is greater than $\epsilon^3 n^2$. Since there are $1/\epsilon^3$ intervals, this would contradict the $n^2$ bound for $\text{tr}(T^2)$.

Thus, we have a partition of $T$ into three matrices:

$$T = T_1 + T_2 + T_3,$$

where $T_1$ is the “structured” component

$$T_1 := \sum_{i < J} \lambda_i u_i u_i^*; \tag{3}$$

and $T_2$ is the “error” component

$$T_2 := \sum_{J \leq i < F(J)} \lambda_i u_i u_i^*; \tag{4}$$

and $T_3$ is the “pseudorandom” component

$$T_3 := \sum_{i \geq F(J)} \lambda_i u_i u_i^*. \tag{5}$$

We will partition the vertex set so that $T_1$ is approximately constant on most clusters. The number of such clusters will be $O_{\epsilon, \epsilon}(1)$. For each $j < J$ we define a partition into clusters on which entry $u_i$ (a complex number) varies by $\frac{\epsilon}{j} n^{-1/2}$. There is also an exceptional cluster of size $\frac{\epsilon}{j} n$ which comes from the vertices for which the entry of $u_i$ is large in magnitude. That is, either its real or its imaginary part is larger than $\sqrt{\frac{\epsilon}{j} n^{-1/2}}$ in absolute value.

To see this, simply place a vertex into the exceptional cluster if the corresponding entry of $u_i$ has the absolute value of either its real or imaginary part at least $\sqrt{\frac{\epsilon}{j} n^{-1/2}}$. Since $\|u_i\|_2 = 1$, this means there can be at most $\frac{\epsilon}{j} n$ such entries. Partition the square of length $2\sqrt{\frac{\epsilon}{j} n^{-1/2}}$ centered at the origin of the complex plane into subsquares of side length $\frac{\epsilon^{3/2}}{j^{3/2}} n^{-1/2}$. There are \(\left(2\sqrt{\frac{\epsilon}{j} n^{-1/2}}\right)^2 / \left(\frac{\epsilon^{3/2}}{j^{3/2}} n^{-1/2}\right)^2 = 4J^4/\epsilon^4\) such subsquares. Partition the vertices according to where its corresponding entry of $u_i$ lies.
Take the union of all vertices in the exceptional clusters and the corresponding exceptional cluster is of size at most \((J-1) \cdot \frac{3}{J} < c\). For the rest of the vertices, we take the common refinement of the partitions defined by each \(u_i, i < J\). This defines a partition of the vertex set \(V = V_0 + V_1 + \cdots + V_M\) in which \(V_0\) is the exceptional set. For \(i = 1, \ldots, M\), the entries over \(V_i\) of each of \(u_1, \ldots, u_{J-1}\) have magnitude at most

\[
\sqrt{2} \cdot 2 \sqrt{\frac{J}{\epsilon} n^{-1/2}},
\]

and differ in magnitude by at most

\[
\sqrt{2} \frac{\epsilon^{3/2}}{J^{3/2}} n^{-1/2}.
\]

Moreover,

\[
M \leq \left( \frac{4J^4}{\epsilon^4} \right)^J.
\]

Now, let \(i, j \in \{1, \ldots, M\}\). We will show that the values of \(T_1\) over the block \(V_i \times V_j\) differ by at most \(8\epsilon\). To see this, let \(a, c \in V_i\) and \(b, d \in V_j\). Then

\[
(T_1)_{ab} - (T_1)_{cd} = \left| \sum_{i < J} \lambda_i u_i(a)u_i(b) - \lambda_i u_i(c)u_i(d) \right|
\]

\[
\leq \sum_{i < J} \left| \lambda_i \left( u_i(a)u_i(b) - u_i(c)u_i(b) + u_i(c)u_i(b) - u_i(c)u_i(d) \right) \right|
\]

\[
\leq \sum_{i < J} n \left| u_i(b) \right| \left| u_i(a) - u_i(c) \right| + n \left| u_i(c) \right| \left| u_i(b) - u_i(d) \right|
\]

\[
\leq J \left( n \cdot 2 \sqrt{2} \sqrt{\frac{J}{\epsilon} n^{-1/2}} \cdot \sqrt{2} \frac{\epsilon^{3/2}}{J^{3/2}} n^{-1/2} + n \cdot 2 \sqrt{2} \sqrt{\frac{J}{\epsilon} n^{-1/2}} \cdot \sqrt{2} \frac{\epsilon^{3/2}}{J^{3/2}} n^{-1/2} \right)
\]

\[
= 2J \left( n \cdot 2 \sqrt{2} \sqrt{\frac{J}{\epsilon} n^{-1/2}} \cdot \sqrt{2} \frac{\epsilon^{3/2}}{J^{3/2}} n^{-1/2} \right)
\]

\[
< 8\epsilon.
\]

As a result, we can conclude that, if \(d_{ij}\) is the mean of the entries in the block \(V_i \times V_j\), then by the triangle inequality,

\[
\left| (T_1)_{ab} - d_{ij} \right| < 16\epsilon.
\]

Next we consider \(T_2\) and observe that \(\text{tr}(T_2^2) = \sum_{J \leq j \leq F(j)} \lambda^2_i < \epsilon^3 n^2\). So, \(\sum_{a, b \in V} \left| (T_2)_{ab} \right|^2 < \epsilon^3 n^2\). Define \(\Sigma_1\) so that for every \((i, j) \notin \Sigma_1\),

\[
\sum_{a \in V_i} \sum_{b \in V_j} \left| (T_2)_{ab} \right|^2 < \epsilon \left| V_i \right| \left| V_j \right|.
\]

Thus,

\[
\epsilon^2 \sum_{(i, j) \in \Sigma_1} \left| V_i \right| \left| V_j \right| \leq \sum_{(i, j) \in \Sigma_1} \sum_{a \in V_i} \sum_{b \in V_j} \left| (T_2)_{ab} \right|^2 \leq \epsilon^3 n^2.
\]
consequently,
\[ \sum_{(i,j) \in \Sigma} |V_i||V_j| \leq \epsilon n^2. \]

Finally, we turn our attention to \( T_3 \). The maximum eigenvalue of \( T_3 \) is \( |\lambda_{F(J)}| \leq n/\sqrt{F(J)} \).
We want to establish that \( n^2/\sqrt{F(J)} \leq \epsilon |V_i||V_j| \) for \((i,j) \notin \Sigma \). Because \( |V_i|, |V_j| \geq \epsilon n/M \), it is sufficient to show that \( F(J) \geq M^4/\epsilon^6 \) because that would verify that
\[ \frac{n^2}{\sqrt{F(J)}} \leq \epsilon |V_i||V_j|. \]

By (8), \( M \leq (4J^4/\epsilon^4)^J \). So the function that suffices is
\[ F(x) \geq \frac{1}{\epsilon^6} \left( \frac{4x^4}{\epsilon^4} \right)^{4x}. \]

Let \( \Sigma \) be the pairs \((i,j) \in \{0,1,\ldots,M\} \) such that either \((i,j) \in \Sigma_1, i = 0, j = 0 \) or \( \min(|V_i|, |V_j|) \leq \frac{\epsilon n}{M} \). Thus,
\[
\sum_{(i,j) \in \Sigma} |V_i||V_j| \leq \sum_{(i,j) \in \Sigma_1} |V_i||V_j| + 2|V_0||V| + 2 \sum_{|V_i| < \epsilon n/M} |V_i||V|
\]
\[
\leq \epsilon n^2 + 2\epsilon n \cdot n + 2M\frac{\epsilon n}{M} \leq 5\epsilon n^2.
\]

Note that in this proof we use coefficients of \( 1q6\epsilon \) and \( 5\epsilon \) in (9) and (11), respectively. We can, of course, choose \( \epsilon/16 \) rather than \( \epsilon \) but we chose these parameters to make the computations somewhat more transparent.

\[ \square \]

3. Spectral version

Lemma 2 (Szemerédi’s regularity lemma, spectral version). Let \( T \) be a self-adjoint \( n \times n \) matrix such that \( \text{tr}(T^2) \leq n^2 \). Let \( V \) be the set of \( n \) indices and let \( \epsilon > 0 \). Then there exists a partition \( V = V_1 \cup \cdots \cup V_M \) for some \( M \leq M(\epsilon) \) with the property that, for all pairs \((i,j) \in \{1,\ldots,M\}^2 \) outside of an exceptional set \( \Sigma \), one has
\[
|v_B^*(T - d_{ij}I)v_A| \leq \epsilon |V_i||V_j|
\]
whenever \( \text{supp}(v_A) \subset V_i, \|v_A\|_2^2 \leq |V_i|, \text{supp}(v_B) \subset V_j \) and \( \|v_B\|_2^2 \leq |V_j| \), for some real number \( d_{ij} \). Furthermore, we have
\[
\sum_{(i,j) \in \Sigma} |V_i||V_j| \leq \epsilon n^2.
\]

Proof. Enumerate \( V = \{1,\ldots,n\} \). Since \( T \) is self-adjoint, it has an eigenvalue decomposition
\[ T = \sum_{i=1}^{n} \lambda_i u_i u_i^*, \]
for some orthonormal basis \( u_1, \ldots, u_n \) of \( \mathbb{C}^n \) (where the vectors are column vectors) and real eigenvalues \( \lambda_1, \ldots, \lambda_n \in \mathbb{R} \). We arrange them in decreasing order of magnitude
\[ |\lambda_1| \geq \cdots \geq |\lambda_n|. \]
In a self-adjoint matrix, the trace of $T^2$ is the sum of the squares of the eigenvalues of $T$ and so $\text{tr}(T^2) = \sum_{j=1}^{n} |\lambda_j|^2$. So we can bound the eigenvalues by observing that $i\lambda_i^2 \leq \sum_{j=1}^{i} \lambda_j^2 \leq n^2$ and so, for all $i \in \{1, \ldots, n\}$,

$$|\lambda_i| \leq \frac{n}{\sqrt{i}}. \tag{14}$$

We will be given a function $F : \mathbb{N} \to \mathbb{N}$ that we will specify later. This function does depend on $\epsilon$ (we suppress this in the notation) and it satisfies the inequality $F(i) > i$ for all integers $i$. We want to find an integer $J$ for which

$$\sum_{J \leq j < F(J)} |\lambda_j|^2 \leq \epsilon^3 n^2. \tag{15}$$

To do this, we consider the partition of $\{1, \ldots, n\}$ into intervals $[F^{(k-1)}(1), F^{(k)}(1) - 1]$ from $k = 1, \ldots, 1/\epsilon^3$ where $F^{(k)}$ represents the $k$th composition of $F$ with itself. Note that either we find a $J = F^{(k-1)}(1)$ for which (15) is satisfied or the sum of $|\lambda_j|^2$ for all $j$ in some interval is greater than $\epsilon^3 n^2$. Since there are $1/\epsilon^3$ intervals, this would contradict the $n^2$ bound for $\text{tr}(T^2)$.

Thus, we have a partition of $T$ into three matrices:

$$T = T_1 + T_2 + T_3,$$

where $T_1$ is the “structured” component

$$T_1 := \sum_{i < J} \lambda_i u_i u_i^*; \tag{16}$$

and $T_2$ is the “error” component

$$T_2 := \sum_{J \leq i < F(J)} \lambda_i u_i u_i^*; \tag{17}$$

and $T_3$ is the “pseudorandom” component

$$T_3 := \sum_{i \geq F(J)} \lambda_i u_i u_i^*. \tag{18}$$

We will partition the vertex set so that $T_1$ is approximately constant on most clusters. The number of such clusters will be $O_{J, \epsilon}(1)$. For each $j < J$ we define a partition into clusters on which entry $u_i$ (a complex number) varies by $\frac{\epsilon}{\sqrt{J}} n^{-1/2}$. There is also an exceptional cluster of size $\frac{\epsilon}{\sqrt{J}} n$ which comes from the vertices for which the entry of $u_i$ is large in magnitude. That is, either its real or its imaginary part is larger than $\epsilon^3 n^{-1/2}$ in absolute value.

To see this, simply place a vertex into the exceptional cluster if the corresponding entry of $u_i$ has the absolute value of either its real or imaginary part at least $\epsilon \sqrt{J} n^{-1/2}$. Since $\|u_i\|_2 = 1$, this means there can be at most $\frac{\epsilon}{\sqrt{J}} n$ such entries. Partition the square of length $2 \sqrt{\frac{J}{\epsilon}} n^{-1/2}$ centered at the origin of the complex plane into subsquares of side length $\frac{\epsilon^{3/2}}{\sqrt{J^{3/2}}} n^{-1/2}$. There are $\left(2 \sqrt{\frac{J}{\epsilon}} n^{-1/2}\right)^2 / \left(\frac{\epsilon^{3/2}}{\sqrt{J^{3/2}}} n^{-1/2}\right)^2 = 4J^4/\epsilon^4$ such subsquares. Partition the vertices according to where its corresponding entry of $u_i$ lies.
Take the union of all vertices in the exceptional clusters and the corresponding exceptional cluster is of size at most \((J - 1) \cdot \frac{\epsilon n}{J} < cn\). For the rest of the vertices, we take the common refinement of the partitions defined by each \(u_i, i < J\). This defines a partition of the vertex set \(V = V_0 + V_1 + \cdots + V_M\) in which \(V_0\) is the exceptional set. For \(i = 1, \ldots, M\), the entries over \(V_i\) of each of \(u_1, \ldots, u_{J-1}\) have magnitude at most

\[
\sqrt{2} \cdot \sqrt{\frac{J}{\epsilon}} n^{-1/2},
\]

and differ in magnitude by at most

\[
\sqrt{2} \frac{\epsilon^{3/2}}{J^{3/2}} n^{-1/2}.
\]

Moreover,

\[
M \leq \left( \frac{4J^4}{\epsilon} \right)^J.
\]

Now, let \(i, j \in \{1, \ldots, M\}\). We will show that the values of \(T_1\) over the block \(V_i \times V_j\) differ by at most \(8\epsilon\). To see this, let \(a, c \in V_i\) and \(b, d \in V_j\). Then

\[
(T_{1})_{ab} - (T_{1})_{cd} = \left| \sum_{i<J} \lambda_i u_i(a) u_i(b) - \sum_{i<J} \lambda_i u_i(c) u_i(d) \right|
\leq \left| \sum_{i<J} |\lambda_i| |u_i(a) u_i(b) - u_i(c) u_i(b) + u_i(c) u_i(b) - u_i(c) u_i(d)| \right|
\leq \sum_{i<J} n|u_i(b)||u_i(a) - u_i(c)| + n|u_i(c)||u_i(b) - u_i(d)|
\leq J \left( n \cdot 2 \sqrt{2} \sqrt{\frac{J}{\epsilon}} n^{-1/2} \cdot \sqrt{2} \frac{\epsilon^{3/2}}{J^{3/2}} n^{-1/2} + n \cdot 2 \sqrt{2} \sqrt{\frac{J}{\epsilon}} n^{-1/2} \cdot \sqrt{2} \frac{\epsilon^{3/2}}{J^{3/2}} n^{-1/2} \right)
= 2J \left( n \cdot 2 \sqrt{2} \sqrt{\frac{J}{\epsilon}} n^{-1/2} \cdot \sqrt{2} \frac{\epsilon^{3/2}}{J^{3/2}} n^{-1/2} \right)
< 8\epsilon.
\]

As a result, we can conclude that, if \(d_{ij}\) is the mean of the entries in the block \(V_i \times V_j\), then by the triangle inequality and Cauchy-Schwarz,

\[
|v_B^* (T_1 - d_{ij} I) v_A| \leq \sum_{a \in V_i} \sum_{b \in V_j} |(T_{1})_{ab} - d_{ij}||v_A(a)||v_B(b)|
< 8\epsilon ||v_A||_1 ||v_B||_1
\leq 8\epsilon |V_i||V_j|.
\]

The last step follows from a basic vector norm inequality which gives \(||v_A||_1 \leq \sqrt{|V_i||v_A||_2 \leq |V_i|\) and \(||v_B||_1 \leq \sqrt{|V_j||v_B||_2 \leq |V_j|\).

Next we consider \(T_2\) and observe that \(\text{tr}(T_2^2) = \sum_{j \leq j \leq F(J)} \lambda_j^2 \leq \epsilon^3 n^2\).
So, $\sum_{a,b \in V} |(T_2)_{ab}|^2 \leq \epsilon^3 n^2$. Define $\Sigma_1$ so that for every $(i, j) \not\in \Sigma_1$,

$$\sum_{a \in V_i} \sum_{b \in V_j} |(T_2)_{ab}|^2 \leq \epsilon |V_i||V_j|.$$  \hspace{1cm} (22)

Thus,

$$\epsilon^2 \sum_{(i, j) \in \Sigma_1} |V_i||V_j| \leq \sum_{(i, j) \in \Sigma_1} \sum_{a \in V_i} \sum_{b \in V_j} |(T_2)_{ab}|^2 \leq \epsilon^3 n^2.$$  \hspace{1cm} (23)

consequently,

$$\sum_{(i, j) \in \Sigma_1} |V_i||V_j| \leq \epsilon n^2.$$  \hspace{1cm} (24)

So, for any $(i, j) \not\in \Sigma_1$, use the fact that $\sum_{a \in V_i} \sum_{b \in V_j} |(T_2)_{ab}|^2 \leq \epsilon |V_i||V_j|$ we use Cauchy-Schwarz to obtain the following bound

$$|v_B^* T_2 v_A|^2 = \left| \sum_{a \in V_i} \sum_{b \in V_j} (T_2)_{ab} v_A(a) \cdot v_B(b) \right|^2 \leq \left( \sum_{a \in V_i} \sum_{b \in V_j} |(T_2)_{ab}|^2 \right) \left( \sum_{a \in V_i} \sum_{b \in V_j} |v_A(a)|^2 |v_B(b)|^2 \right) = (\epsilon^2 |V_i||V_j|) \|v_A\|^2 \|v_B\|^2.$$  \hspace{1cm} (25)

The last step follows from $\|v_A\|^2 \leq |V_i|$ and $\|v_B\|^2 \leq |V_j|$.

Finally, we turn our attention to $T_3$. Let $v_A$ and $v_B$ be vectors such that $\|v_A\|^2, \|v_B\|^2 \leq n$. Since the maximum eigenvalue of $T_3$ is $|\lambda_{F(J)}| \leq n/\sqrt{F(J)}$, we have, first by Cauchy-Schwarz,

$$|v_B^* T_3 v_A| \leq |\lambda_{F(J)}|\|v_A\|_2\|v_B\|_2 \leq n^2/\sqrt{F(J)}.$$  \hspace{1cm} (26)

Let $\Sigma$ be the pairs $(i, j) \in \{0, 1, \ldots, M\}$ such that either $(i, j) \in \Sigma_1$, $i = 0$, $j = 0$ or $\min(|V_i|, |V_j|) \leq \frac{\epsilon n}{M}$. Thus,

$$\sum_{(i, j) \in \Sigma} |V_i||V_j| \leq \sum_{(i, j) \in \Sigma_1} |V_i||V_j| + 2|V_0||V| + 2 \sum_{|V_i| < \epsilon n/M} |V_i||V| \leq \epsilon n^2 + 2\epsilon n \cdot n + 2M \frac{\epsilon n}{M} n \leq 5\epsilon n^2.$$  \hspace{1cm} (27)

If $(i, j) \not\in \Sigma$, then for all $A \subseteq V_i$ and $B \subseteq V_j$,

$$|v_B^*(T - d_{ij})v_A| \leq |v_B^*(T_1 - d_{ij})v_A| + |v_B^* T_2 v_A| + |v_B^* T_3 v_A| \leq 8\epsilon |V_i||V_j| + \epsilon |V_i||V_j| + n^2/\sqrt{F(J)}.$$  \hspace{1cm} (28)

Finally, we want to establish that $n^2/\sqrt{F(J)} \leq \epsilon |V_i||V_j|$ for $(i, j) \not\in \Sigma$. This would establish that

$$|v_B^*(T - d_{ij})v_A| \leq 10\epsilon |V_i||V_j|.$$

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Because $|V_i|,|V_j| \geq \epsilon n/M$, it is sufficient to show that $F(J) \geq M^4/\epsilon^6$ because that would verify that
\[
\frac{n^2}{\sqrt{F(J)}} \leq \epsilon^3 n^2 M^2 \leq \epsilon |V_i||V_j|.
\]

By (21), $M \leq (4J^4/\epsilon^4)^J$. So the function that suffices is
\[
F(x) \geq \frac{1}{\epsilon^6} \left(\frac{4x^4}{\epsilon^4}\right)^{4x}.
\]

Note that in this proof we use coefficients of $10\epsilon$ and $5\epsilon$ in (24) and (23), respectively. We can, of course, choose $\epsilon/10$ rather than $\epsilon$ but we chose these parameters to make the computations somewhat more transparent. \qed 

4. Graph version

**Lemma 3** (Szemerédi’s regularity lemma, spectral version). Let $G = (V, E)$ be a graph on $n$ vertices and let $\epsilon > 0$. Then there exists a partition $V = V_1 \cup \cdots \cup V_M$ for some $M \leq M(\epsilon)$ with the property that, for all pairs $(i, j) \in \{1, \ldots, M\}^2$ outside of an exceptional set $\Sigma$, one has
\[
|E(A, B) - d_{ij}|A||B|| \ll \epsilon |V_i||V_j|
\]
whenever $A \subset V_i$, $B \subset V_j$, for some real number $d_{ij}$, where
\[
E(A, B) := \{|(a, b) \in A \times B : \{a, b\} \in E\|
\]
is the number of edges between $A$ and $B$. Furthermore, we have
\[
\sum_{(i,j) \in \Sigma} |V_i||V_j| \ll \epsilon |V|^2.
\]

**Proof.** Here we do the proof directly, even though this is a direct consequence of the matrix version above.

Enumerate $V = \{1, \ldots, n\}$. Let $T$ be the incidence matrix of $G$ and note that since $T$ is self-adjoint, it has an eigenvalue decomposition
\[
T = \sum_{i=1}^n \lambda_i u_i u_i^*,
\]
for some orthonormal basis $u_1, \ldots, u_n$ of $\mathbb{C}^n$ (where the vectors are column vectors) and real eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. We arrange them in decreasing order of magnitude
\[
|\lambda_1| \geq \cdots \geq |\lambda_n|.
\]

In a self-adjoint matrix, the trace of $T^2$ is the sum of the squares of the eigenvalues of $T$ and so $\text{tr}(T^2) = \sum_{j=1}^n |\lambda_j|^2$. In addition, it is the sum of the degrees of degrees of the graph, hence $\text{tr}(T^2) = 2|E(G)| \leq n^2$. So we can bound the eigenvalues by observing that $i\lambda_i^2 \leq \sum_{j=1}^i \lambda_j^2 \leq n^2$ and so, for all $i \in \{1, \ldots, n\},$
\[
|\lambda_i| \leq \frac{n}{\sqrt{i}}.
\]
We will be given a function $F : \mathbb{N} \rightarrow \mathbb{N}$ that we will specify later. This function does depend on $\epsilon$ (we suppress this in the notation) and it satisfies the inequality $F(i) > i$ for all integers $i$. We want to find an integer $J$ for which

$$
\sum_{J \leq j < F(J)} |\lambda_j|^2 \leq \epsilon^3 n^2.
$$

(28)

To do this, we consider the partition of $\{1, \ldots, n\}$ into intervals $[F^{(k-1)}(1), F^{(k)}(1) - 1]$ from $k = 1, \ldots, 1/\epsilon^3$ where $F^{(k)}$ represents the $k^{th}$ composition of $F$ with itself. Note that either we find a $J = F^{(k-1)}(1)$ for which (28) is satisfied or the sum of $|\lambda_j|^2$ for all $j$ in some interval is greater than $\epsilon^3 n^2$. Since there are $1/\epsilon^3$ intervals, this would contradict the $n^2$ bound for $\text{tr}(T^2)$.

Thus, we have a partition of $T$ into three matrices:

$$
T = T_1 + T_2 + T_3,
$$

where $T_1$ is the “structured” component

$$
T_1 := \sum_{i < J} \lambda_i u_i u_i^*,
$$

(29)

and $T_2$ is the “error” component

$$
T_2 := \sum_{J \leq i < F(J)} \lambda_i u_i u_i^*,
$$

(30)

and $T_3$ is the “pseudorandom” component

$$
T_3 := \sum_{i \geq F(J)} \lambda_i u_i u_i^*.
$$

(31)

We will partition the vertex set so that $T_1$ is approximately constant on most clusters. The number of such clusters will be $O_{\epsilon, \epsilon}(1)$. For each $j < J$ we define a partition into clusters on which entry $u_i$ (a complex number) varies by $\frac{\epsilon}{2} n^{1/2}$. There is also an exceptional cluster of size $\frac{\epsilon}{2} n$ which comes from the vertices for which the entry of $u_i$ is large in magnitude. That is, either its real or its imaginary part is larger than $\sqrt{\frac{\epsilon}{2} n^{1/2}}$ in absolute value.

To see this, simply place a vertex into the exceptional cluster if the corresponding entry of $u_i$ has the absolute value of either its real or imaginary part at least $\sqrt{\frac{\epsilon}{2} n^{1/2}}$. Since $\|u_i\|_2 = 1$, this means there can be at most $\frac{\epsilon}{2} n$ such entries. Partition the square of length $2\sqrt{\frac{\epsilon}{2} n^{1/2}}$ centered at the origin of the complex plane into subsquares of side length $\frac{\epsilon^{3/2}}{\sqrt{3/2}} n^{-1/2}$. There are $\left(2\sqrt{\frac{\epsilon}{2} n^{1/2}}\right)^2 / \left(\frac{\epsilon^{3/2}}{\sqrt{3/2}} n^{-1/2}\right)^2 = 4J^2/\epsilon^4$ such subsquares. Partition the vertices according to where its corresponding entry of $u_i$ lies.

Take the union of all vertices in the exceptional clusters and the corresponding exceptional cluster is of size at most $(J - 1) \cdot \frac{\epsilon}{2} n < \epsilon n$. For the rest of the vertices, we take the common refinement of the partitions defined by each $u_i$, $i < J$. This defines a partition of the vertex set $V = V_0 + V_1 + \cdots + V_M$ in which $V_0$ is the exceptional set. For $i = 1, \ldots, M$, the entries
over $V_i$ of each of $u_1, \ldots, u_{J-1}$ have magnitude at most

$$\sqrt{2} \cdot 2 \sqrt{\frac{J}{\epsilon}} n^{-1/2}. \quad (32)$$

and differ in magnitude by at most

$$\sqrt{2} \frac{\epsilon^{3/2}}{\sqrt{J}3/2} n^{-1/2}. \quad (33)$$

Moreover,

$$M \leq \left( \frac{4J^4}{\epsilon^4} \right)^{J}. \quad (34)$$

Now, let $i, j \in \{1, \ldots, M\}$. We will show that the values of $T_1$ over the block $V_i \times V_j$ differ by at most $8\epsilon$. To see this, let $a, c \in V_i$ and $b, d \in V_j$. Then

$$(T_1)_{ab} - (T_1)_{cd} = \left| \sum_{i<J} \lambda_i \mathbf{u}_i(a) \mathbf{u}_i(b) - \lambda_i \mathbf{u}_i(c) \mathbf{u}_i(d) \right|$$

$$\leq \sum_{i<J} |\lambda_i| |\mathbf{u}_i(a) \mathbf{u}_i(b) - \mathbf{u}_i(c) \mathbf{u}_i(b) + \mathbf{u}_i(c) \mathbf{u}_i(b) - \mathbf{u}_i(c) \mathbf{u}_i(d)|$$

$$\leq \sum_{i<J} n|\mathbf{u}_i(b)||\mathbf{u}_i(a) - \mathbf{u}_i(c)| + n|\mathbf{u}_i(c)||\mathbf{u}_i(b) - \mathbf{u}_i(d)|$$

$$\leq J \left( n \cdot 2\sqrt{2} \sqrt{\frac{J}{\epsilon}} n^{-1/2} \cdot \sqrt{2} \frac{\epsilon^{3/2}}{\sqrt{J}3/2} n^{-1/2} + n \cdot 2\sqrt{2} \sqrt{\frac{J}{\epsilon}} n^{-1/2} \cdot \sqrt{2} \frac{\epsilon^{3/2}}{\sqrt{J}3/2} n^{-1/2} \right)$$

$$= 2J \left( n \cdot 2\sqrt{2} \sqrt{\frac{J}{\epsilon}} n^{-1/2} \cdot \sqrt{2} \frac{\epsilon^{3/2}}{\sqrt{J}3/2} n^{-1/2} \right)$$

$$< 8\epsilon.$$

As a result, we can conclude that, if $d_{ij}$ is the mean of the entries in the block $V_i \times V_j$, then by the triangle inequality and Cauchy-Schwarz,

$$|\mathbf{1}_B(T_1 - d_{ij}\mathbf{1})\mathbf{1}_A| \leq \sum_{a \in A} \sum_{b \in B} |(T_1)_{ab} - d_{ij}|$$

$$< 8\epsilon|A||B|$$

$$\leq 8\epsilon|V_i||V_j|.$$ 

Next we consider $T_2$ and observe that $\text{tr}(T_2^2) = \sum_{J \leq J \leq F(J)} \lambda_i^2 \leq \epsilon^3 n^2$. So, $\sum_{a,b \in V} |(T_2)_{ab}|^2 \leq \epsilon^3 n^2$. Define $\Sigma_1$ so that for every $(i, j) \notin \Sigma_1$,

$$\sum_{a \in V_i} \sum_{b \in V_j} |(T_2)_{ab}|^2 \leq \epsilon|V_i||V_j|. \quad (35)$$

Thus,

$$\epsilon^2 \sum_{(i,j) \in \Sigma_1} |V_i||V_j| \leq \sum_{(i,j) \in \Sigma_1} \sum_{a \in V_i} \sum_{b \in V_j} |(T_2)_{ab}|^2 \leq \epsilon^3 n^2.$$
consequently,
\[ \sum_{(i,j) \in \Sigma} |V_i||V_j| \leq \epsilon n^2. \]

So, for any \((i, j) \not\in \Sigma_1\), use the fact that \(\sum_{a \in V_i} \sum_{b \in V_j} |(T_2)_{ab}|^2 \leq \epsilon |V_i||V_j|\) we use Cauchy-Schwarz to obtain the following bound

\[ |1_B^* T_2 1_A|^2 = \left| \sum_{a \in A} \sum_{b \in B} (T_2)_{ab} \right|^2 \leq \left( \sum_{a \in A} \sum_{b \in B} |(T_2)_{ab}|^2 \right) |A||B| \]
\[ = (\epsilon^2 |V_i||V_j|) |A||B| \]
\[ \sum_{(i,j) \in \Sigma} |V_i||V_j| \leq \sum_{(i,j) \in \Sigma_1} |V_i||V_j| + 2|V_0||V| + 2 \sum_{|V_i| < \epsilon n/M} |V_i||V| \]
\[ \leq \epsilon n^2 + 2\epsilon n \cdot n + 2M \frac{\epsilon n}{M} n \leq 5\epsilon n^2. \]

Finally, we turn our attention to \(T_3\). Since the maximum eigenvalue of \(T_3\) is \(|\lambda_{F(J)}| \leq n/\sqrt{F(J)}\), we have, first by Cauchy-Schwarz,

\[ |1_B^* T_3 1_A| \leq |\lambda_{F(J)}||A||B| \leq n^2/\sqrt{F(J)}. \]

Let \(\Sigma\) be the pairs \((i, j) \in \{0, 1, \ldots, M\}\) such that either \((i, j) \in \Sigma_1\), \(i = 0\), \(j = 0\) or \(\min(|V_i|, |V_j|) \leq \frac{\epsilon n}{M}\). Thus,

\[ \sum_{(i,j) \in \Sigma} |V_i||V_j| \leq \sum_{(i,j) \in \Sigma_1} |V_i||V_j| + 2|V_0||V| + 2 \sum_{|V_i| < \epsilon n/M} |V_i||V| \]
\[ \leq \epsilon n^2 + 2\epsilon n \cdot n + 2M \frac{\epsilon n}{M} n \leq 5\epsilon n^2. \]

If \((i, j) \not\in \Sigma\), then for all \(A \subseteq V_i\) and \(B \subseteq V_j\),

\[ |1_B^*(T - d_{ij}) 1_A| \leq |1_B^*(T_1 - d_{ij} 1_A| + |1_B^* T_2 1_A| + |1_B^* T_3 1_A| \]
\[ \leq 8\epsilon |V_i||V_j| + \epsilon |V_i||V_j| + n^2/\sqrt{F(J)}. \]

Finally, we want to establish that \(n^2/\sqrt{F(J)} \leq \epsilon |V_i||V_j|\) for \((i, j) \not\in \Sigma\). This would establish that

\[ (37) \quad |1_B^*(T - d_{ij}) 1_A| \leq 10\epsilon |V_i||V_j|. \]

Because \(|V_i|, |V_j| \geq \epsilon n/M\), it is sufficient to show that \(F(J) \geq M^4/\epsilon^6\) because that would verify that

\[ \frac{n^2}{\sqrt{F(J)}} \leq \frac{\epsilon^3 n^2}{M^2} \leq \epsilon |V_i||V_j|. \]

By (34), \(M \leq (4J^4/\epsilon^4)^J\). So the function that suffices is

\[ F(x) \geq \frac{1}{\epsilon^6} \left( \frac{4x^4}{\epsilon^4} \right)^{4x} \]

Note that in this proof we use coefficients of 10\(\epsilon\) and 5\(\epsilon\) in (37) and (36), respectively. We can, of course, choose \(\epsilon/10\) rather than \(\epsilon\) but we chose these parameters to make the
computations somewhat more transparent.

**REFERENCES**