Do AT LEAST THREE (3) of the following:

1. Use the symmetric version of the Lovász Local Lemma to prove that 
   \( R(k,k) > \frac{\sqrt{2}}{2} (1 - o(1)) k^{2k/2} \). This is an improvement of a multiplicative factor of 2 over the naive probabilistic bound in Erdős’ 1947 paper.

2. Prove the Slicing Lemma for \( \epsilon \)-regular pairs:

   **Lemma (Slicing Lemma)** Let \((A, B)\) be an \( \epsilon \)-regular pair with density \( d \). If \( A' \subseteq A \) and \( B' \subseteq B \) such that \(|A'| \geq \epsilon |A|\) and \(|B'| \geq \epsilon |B|\), then \((A', B')\) is an \( \epsilon' \)-regular pair, with \( \epsilon' = \max \left\{ 2\epsilon, \frac{|A|}{|A'|}, \frac{|B|}{|B'|} \epsilon \right\} \) and density in \([d - \epsilon, d + \epsilon]\).

3. Consider the following definition:

   **Definition.** A pair \((A, B)\) is said to be \((\epsilon, \delta)\)-super-regular if the following occur:
   - \((A, B)\) is \( \epsilon \)-regular,
   - \( \deg_B(a) \geq \delta |B| \), for all \( a \in A \) and
   - \( \deg_A(b) \geq \delta |A| \), for all \( b \in B \).

   Prove the following proposition:

   **Proposition.** Let \( \epsilon < 1/2 \) and \( d > 2\epsilon \). Let \((A, B)\) be an \( \epsilon \)-regular pair with density \( d \), \(|A| = |B| = L\). Then there exist \( A' \subseteq A \) and \( B' \subseteq B \) such that
   - \((A', B')\) is \((2\epsilon, d - 2\epsilon)\)-super-regular,
   - \(|A'| = |B'| = \lceil (1 - \epsilon) L \rceil\), and
   - \( d(A', B') \in [d - \epsilon, d + \epsilon] \).

4. The Crowdsource Problem: For this problem, you can work together and discuss the problem with your fellow students. You must write up your own solution, however.

   An arithmetic progression of length \( k \) is a sequence \( a, a + d, a + 2d, \ldots, a + (k - 1)d \). The van der Waerden number \( W(k) \) is the least \( n \) so that if
\{1, \ldots, n\} is two-colored, then it has a monochromatic arithmetic progression of length \(k\).

Use the Lovász Local Lemma to prove that \(W(k) > (1 - o(1))2^{k-1} \frac{2^k - 1}{ek}\).