M608 (S12) Homework 2

Due date: 16 February 2012

Do ALL 3 of the following:

1. Prove the following for real-valued random variables \( X, X_1, \ldots, X_n \)
   - \( \mathbb{E}[X^2] \geq (\mathbb{E}[X])^2 \).
   - \( \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] \)
   - If \( X_1, \ldots, X_n \) are pairwise independent, each with finite mean and variance, then
     \[
     \text{Var} \left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \text{Var}(X_i).
     \]
   - If \( X_1, \ldots, X_n \) are mutually independent, then
     \[
     \mathbb{E} \left[ \prod_{i=1}^{n} X_i \right] = \prod_{i=1}^{n} \mathbb{E}[X_i].
     \]
     (You only need to prove this last one for discrete random variables.)

2. Prove the following:
   - If \( \alpha \) is a positive real number and \( y \) is a real number such that \( |y| \leq 1 \), then
     \[
     e^{\alpha y} \leq \cosh(\alpha) + \sinh(\alpha)y.
     \]  
     Hence, if \( Y \) is a random variable with \( |Y| \leq 1 \), then \( \mathbb{E}[e^{\alpha Y}] \leq \mathbb{E}[(\cosh(\alpha) + \sinh(\alpha)Y) = \cosh(\alpha) + \sinh(\alpha)\mathbb{E}[Y].
     \]
   - If \( x \) is a real number, then \( \cosh(x) \leq e^{x^2/2}. \)

3. The Crowdsource Problem: For this problem, you can work together and discuss the problem with your fellow students. You must write up your own solution, however.
   - If \( G \) is a graph on \( n \) vertices with no isolated vertices, then \( G \) has a dominating set of size at most \( \lfloor n/2 \rfloor \).
   - For any positive integer \( n \) and any \( \delta, 1 \leq \delta \leq n-1 \), construct a simple graph on \( n \) vertices with minimum degree \( \delta \) and no dominating set smaller than \( \left\lfloor \frac{n}{\delta+1} \right\rfloor \).
• For any positive integer $n$ and any even $\delta$, $1 \leq \delta \leq n - 2$, construct a simple graph on $n$ vertices with minimum degree $\delta$ and no dominating set smaller than $2 \left\lfloor \frac{n}{\delta+2} \right\rfloor$. 