M608 (S12) Homework 1

Due date: 31 January 2012

Do any three (3) of #1-#4. Also, do #5:

1. Assuming König’s theorem, prove Hall’s matching theorem.

   **Theorem** (König’s theorem). Let $G$ be a bipartite graph. The maximum size of a matching in $G$ is equal to the minimum size of a vertex cover in $G$.

   **Theorem** (Hall’s matching theorem). Let $G = (A, B; E)$ be a bipartite graph. The graph $G$ has a matching that saturates $A$ if and only if
   \[
   |N(X)| \geq |X| \quad \text{for all} \quad X \subseteq A. \tag{1}
   \]

2. Prove that $T_{n,r}$ is the $n$-vertex $r$-partite graph with the most number of edges.

3. Assuming the original form of Hajnal-Szemerédi, prove the complementary form.

   **Theorem** (Hajnal-Szemerédi). If $G$ is a simple graph on $n$ vertices with maximum degree $\Delta(G) \leq r$, then $G$ has an equitable $(r + 1)$-coloring.

   An equitable $k$-coloring of a graph $G$ is a proper coloring of $G$ in $k$ colors such that any two color classes differ in size by at most 1.

   **Theorem** (Hajnal-Szemerédi – complementary form). If $G$ is a simple graph on $n$ vertices with minimum degree $\delta(G) \geq \frac{k - 1}{k}n$, then $G$ contains a subgraph that consists of $\lfloor n/k \rfloor$ vertex-disjoint copies of $K_k$.

4. The diameter of a graph $G$ is the largest distance between two vertices. Prove that, for integers $r, d \geq 1$, an $r$-regular graph with diameter $d$ has at most $1 + r \sum_{i=0}^{d-1}(r - 1)^i$ vertices.

5. **The Crowdsource Problem**: For this problem, you can work together and discuss the problem with your fellow students. You must write up your own solution, however. Do two of the following three:
(a) A matrix with real nonnegative entries is **doubly stochastic** if the sum of the entries in any row and any column equals one. A **permutation matrix** is a doubly stochastic $(0,1)$-matrix. A matrix $A$ is a **convex combination** of matrices $A_1, \ldots, A_s$ if there exist nonnegative reals $\lambda_1, \ldots, \lambda_s$ such that $\sum_{i=1}^{s} \lambda_i = 1$ and $A = \sum_{i=1}^{s} \lambda_i A_i$.

Use Hall’s matching theorem to prove the Birkhoff-Von Neumann theorem:

**Theorem** (Birkhoff-Von Neumann). Any doubly stochastic matrix can be written as a convex combination of permutation matrices.

**Hint:** An $n \times n$ matrix gives rise naturally to a bipartite graph $G = (A, B; E)$ with $|A| = |B| = n$ and a permutation matrix corresponds to a perfect matching.

(b) A network is a directed graph with a **source** $s$ and a **target** $t$ with each edge assigned an integer called its capacity. An **edge cut** $[S, S']$ is the set of edges directed from $S$ to $S'$. The **value** of an edge cut is the sum of the capacities. A **flow** is a function $f$ on the arcs in which $f(u, v)$ is at most the capacity of $(u, v)$ and we define $f^+(v) = \sum_u f(v, u)$ (flow out of $v$) and $f^-(v) = \sum_u f(u, v)$ (flow into $v$) with the condition that $f^+(v) = f^-(v)$ for all $v \notin \{s, t\}$. The **value of a flow** is $f^+(s) - f^-(s)$.

Prove the Max Flow-Min Cut theorem. You may do so directly (i.e., without Hall).

**Theorem** (Max Flow-Min Cut). The maximum value of a flow in a network $D$ is equal to the value of a minimum cut of $D$.

(c) Using the Max Flow-Min Cut theorem, prove Hall’s theorem.