1. Let $k \geq 2$ and $G = (V_1, \ldots, V_k; E)$ be a $k$-partite graph with $|V_1| = \cdots = |V_k| = n$ and, for each distinct $i, j$ in $\{1, \ldots, k\}$, the bipartite graph $G[V_i, V_j]$ has minimum degree at least $\left(1 - \frac{1}{2(k-1)}\right)n$. Prove that $G$ has a subgraph consisting of $n$ vertex-disjoint copies of $K_k$.

**Solution.** Recall that $G$ is said to be a balanced $k$-partite graph on $kn$ vertices. We proceed by induction on $k$. For the base case of $k = 2$, a balanced bipartite graph on $2n$ vertices with minimum degree $n/2$ obeys Hall’s condition. To see this, let $X \subset V_1$. If $|X| \leq n/2$, then $|N(X)| \geq n/2 \geq |X|$. If $|X| > n/2$, then each vertex in $V_2$ is adjacent to some vertex in $X$ and so $|N(X)| = n \geq |X|$. Thus, there is a perfect matching, which settles the case where $k = 2$.

Suppose the statement is true for $k-1 \geq 2$. Consider $G' \overset{\text{def}}{=} G[V_1, \ldots, V_{k-1}]$. Each of the bipartite graphs $G'[V_i, V_j] = G[V_i, V_j]$ have minimum degree at least $\left(1 - \frac{1}{2(k-1)}\right)n \geq \left(1 - \frac{1}{2(k-2)}\right)n$. By the inductive hypothesis, $G'$ has a subgraph consisting of $n$ vertex-disjoint copies of $K_{k-1}$.

Now, construct an auxiliary bipartite graph $G'' = (W, V_k; E'')$ in which the elements of $W$ are the individual copies of $K_{k-1}$ found in the subgraph above. For $w \in W$ and $v \in V_k$, we say that $wv \in E''$ if, in $G$, each vertex in $w$ is adjacent to $v$. Each member of $W$ is nonadjacent to at most $(k-1)\left(1 - \frac{1}{2(k-1)}\right)n = n/2$ vertices in $V_k$ and each member of $V_k$ is nonadjacent to at most $(k-1)\left(\frac{1}{2(k-1)}\right)n = n/2$ members of $W$. Thus, $G''$ is a balanced bipartite graph on $2n$ vertices with minimum degree at least $n/2$. As we saw in the base case, such a graph satisfies Hall’s condition and so it has a perfect matching, which corresponds to a subgraph that consists of $n$ vertex-disjoint copies of $K_k$. This completes the proof.

2. Let $r, k \geq 2$ and let $H$ be an $r$-uniform hypergraph. We say that two hyperedges “meet” if they have a nonempty intersection. Prove that if every hyperedge meets fewer than $k^{r-1}/e(k-1)^r$ other hyperedges, then the vertices of $H$ can be $k$-colored in such a way that every edge contains...
at least one vertex of each color.

**Solution.** Color the vertices randomly. Let $A_i$ be the event that that edge $i$ fails to receive at least one of the colors. By Boole’s inequality, $\Pr(A_i) \leq k \left( \frac{k-1}{k} \right)^r$. We construct the auxiliary digraph whose vertices are the edges of $H$ and $i \rightarrow j$ iff $i$ meets $j$. It is easy to see that this is a dependency digraph.

In the language of the Lovász local lemma, $p = k \left( \frac{k-1}{k} \right)^r$, $d = k^{r-1}/e(k-1)^r$ and so,

$$ep(d+1) = e \cdot k \left( \frac{k-1}{k} \right)^r \cdot k^{r-1}/e(k-1)^r = 1.$$ 

By the lemma, $\Pr \left( \bigwedge_{i=1}^n A_i \right) > 0$ and $\bigwedge_{i=1}^n \overline{A_i}$ is exactly the event that each hyperedge receives all $k$ colors.

Note, I should have written something like “meets at most $k^{r-1}/e(k-1)^r − 1$” other hyperedges, but I didn’t worry about the “+1” in the Local Lemma expression anyway.

3. Let $f_1(p), f_2(p), \ldots$ be a sequence of lines with the property that $f_i(p) \geq 0$ for all $p \in [0,1]$ and the slope is between $-1$ and 1. Prove that $f(p) \overset{\text{def}}{=} \inf_i \{ f_i(p) \}$ is concave down over $[0,1]$.

(That is, prove that if $p_1, p_2 \in [0,1]$, then $f(tp_1 + (1-t)p_2) \geq tf(p_1) + (1-t)f(p_2)$ for any $t \in [0,1]$.)

**Solution.** Let $i$ be any positive integer. By linearity,

$$tf(p_1) + (1-t)f(p_2) \leq tf_i(p_1) + (1-t)f_i(p_2) = f_i(tp_1 + (1-t)p_2)$$

Because $i$ is arbitrary, $tf(p_1) + (1-t)f(p_2) \leq f_i(tp_1 + (1-t)p_2)$.

4. Let $k \geq 2$ and let $H$ be a graph on vertex set $\{w_1, \ldots, w_k\}$. Let $1 \leq \epsilon \ll d$ and let $G = (V_1, \ldots, V_k; E)$ be a $k$-tuple with $|V_1| = \cdots = |V_k| = L$ and, for each distinct $i, j$ in $\{1, \ldots, k\}$, the pair $(V_i, V_j)$ is $\epsilon$-regular such that

$$d(V_i, V_j) > d + (k-2)d^{r-2}, \quad \text{if } w_i w_j \text{ is an edge in } H;$$
$$d(V_i, V_j) < 1 - d - (k-2)d^{r-2}, \quad \text{if } w_i w_j \text{ is a nonedge in } H.$$

Prove that $G$ has an induced copy of $H$.

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1The expression $\epsilon < \min\{d^{k-1}, 1/(k-1)\}$ suffices and is convenient for a smooth inductive hypothesis. This is also true for the seemingly strange expressions for the density.
Note: You can get up to 60% credit if you prove it in the case where $H = P_3$, the path on 3 vertices.

Solution. We proceed by induction on $k$ where $\epsilon < \min\{d^{k-1}, 1/(k-1)\}$.

In the case where $k = 2$, we have an $\epsilon$-regular pair. If the graph $H$ is an edge then it has density at least $d > 0$ and there is an edge in the pair. If the graph $H$ is a nonedge then it has density at most $1 - d < 1$ and there is a nonedge in the pair.

Suppose the statement is true for $k-1 \geq 2$. That is, suppose the following:

Let $H'$ be a graph on vertex set $\{w_1, \ldots, w_{k-1}\}$. Let $\epsilon' < \min\{d^{k-2}, 1/(k-2)\}$ and let $G' = (V'_1, \ldots, V'_{k-1}; E')$ be a $(k-1)$-tuple with $|V'_1| = \cdots = |V'_{k-1}| = L'$ and, for each distinct $i, j$ in $\{1, \ldots, k-1\}$, the pair $(V'_i, V'_j)$ is $\epsilon'$-regular such that

\[
\begin{align*}
    d(V_i, V_j) &> d + (k-3)\epsilon'd^{-k+3}, \quad \text{if } w_i w_j \text{ is an edge in } H; \\
    d(V_i, V_j) &< 1 - d - (k-3)\epsilon'd^{-k+3}, \quad \text{if } w_i w_j \text{ is a nonedge in } H.
\end{align*}
\]

Then $G'$ has an induced copy of $H'$.

Let $X \subseteq V_k$ be the set of vertices such that $x \in X$ is adjacent to at least $(d + (k-3)\epsilon)L$ vertices in $V_i$ whenever $w_i w_k$ is an edge and is nonadjacent to at least $(d + (k-3)\epsilon)L$ vertices in $V_i$ whenever $w_i w_k$ is a nonedge. By $\epsilon$-regularity, $|X| \geq L - (k-1)\epsilon L > 0$. Hence $X$ is nonempty.

Choose $x \in X \subseteq V_k$. For $i = 1, \ldots, k-1$, choose $V'_i$ to be a set of size $L' \overset{\text{def}}{=} [(d + (k-3)\epsilon)L]$ in $V_i$ such that if $w_i w_k$ is an edge, then $x$ is adjacent to every vertex in $V'_i$ and if $w_i w_k$ is a nonedge, then $x$ is nonadjacent to every vertex in $V'_i$.

By the slicing lemma, each pair $(V'_i, V'_j)$ is $(\epsilon L/L')$-regular. Let

\[
\epsilon' \overset{\text{def}}{=} \epsilon d^{-1} \geq \frac{\epsilon}{d + (k-3)\epsilon} \geq \frac{\epsilon}{L'}.
\]

Since $d + (k-3)\epsilon > \epsilon$, have the property that, for distinct $i, j$ in $\{1, \ldots, k-1\}$,

\[
\begin{align*}
    d(V'_i, V'_j) &> d + (k-2)\epsilon d^{-k+2} - \epsilon, \quad \text{if } w_i w_j \text{ is an edge in } H; \\
    d(V'_i, V'_j) &< 1 - d - (k-2)\epsilon d^{-k+2} + \epsilon, \quad \text{if } w_i w_j \text{ is a nonedge in } H.
\end{align*}
\]

Since $d + (k-2)\epsilon d^{-k+2} - \epsilon > d + (k-3)\epsilon d^{-k+3}$ and $1 - d - (k-2)\epsilon d^{-k+2} + \epsilon < 1 - d - (k-3)\epsilon d^{-k+3}$, we can now apply the inductive hypothesis to obtain our induced copy of $H'$ and, together with vertex $x$, we get the induced copy of $H$. 
