Do at least 4 of the following:

1. Prove the following:
   - If $G$ is a graph on $n$ vertices with no isolated vertices, then $G$ has a dominating set of size at most $\left\lfloor \frac{n}{2} \right\rfloor$.
   - For any positive integer $n$ and any $\delta$, $1 \leq \delta \leq n-1$, construct a simple graph on $n$ vertices with minimum degree $\delta$ and no dominating set smaller than $\left\lfloor \frac{n}{\delta+1} \right\rfloor$.
   - For any positive integer $n$ and any even $\delta$, $1 \leq \delta \leq n-1$, construct a simple graph on $n$ vertices with minimum degree $n$ and no dominating set smaller than $2 \left\lfloor \frac{n}{\delta+2} \right\rfloor$.

2. Prove the following:
   - If $X_1, \ldots, X_n$ are pairwise independent random variables, each with finite mean, then
     \[
     \text{Var} \left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \text{Var}(X_i).
     \]
   - If $X_1, \ldots, X_n$ are mutually independent, discrete random variables, then
     \[
     \mathbb{E} \left[ \prod_{i=1}^{n} X_i \right] = \prod_{i=1}^{n} \mathbb{E}[X_i].
     \]

3. Prove that, for a random variable $X$, \( \mathbb{E}[X^2] \geq (\mathbb{E}[X])^2 \).

4. Prove the following:
   - If $\alpha$ is a positive real number and $y$ is a real number such that $|y| \leq 1$, then
     \[
     e^{\alpha y} \leq \cosh(\alpha) + \sinh(\alpha) y. \tag{1}
     \]
     Hence, if $Y$ is a random variable with $|Y| \leq 1$, then
     \[
     \mathbb{E}[e^{\alpha Y}] \leq \mathbb{E}[\cosh(\alpha) + \sinh(\alpha)Y] = \cosh(\alpha) + \sinh(\alpha)\mathbb{E}[Y].
     \]
• If $x$ is a real number, then $\cosh(x) \leq e^{x^2/2}$.

5. Do the following:

• The minimum rank of an $n$-vertex graph $G$, denoted $\text{mr}(G)$, is the minimum rank over all $n \times n$ matrices $A$ such that, for distinct $i, j$, if $A_{ij} = 0$ then vertex $i$ is nonadjacent to vertex $j$ in $G$. Note that the diagonal entries in $A$ can be arbitrary. A principle minor (of size $n - 1$) of $A$ is obtained by deleting the $i^{th}$ row and $i^{th}$ column of $A$. Prove that, for any principle minor, $A'$ of $A$ that

$$\text{rank}(A) - 2 \leq \text{rank}(A') \leq \text{rank}(A).$$

Use this fact to show that, if $G \sim G(n, p)$ then $\text{mr}(G)$ is tightly concentrated around its mean. Find the proper expression for the probability. It has been recently showed by Bryan Shader that $\mathbb{E}[G(n, 1/2)] \geq n/14$.

• Let $\mathcal{H}$ be an $r$-uniform hypergraph with $m$ hyperedges. Prove that $\mathcal{H}$ has Property B if $m < 2^{r-1}$.

6. Use the symmetric version of the Lovász Local Lemma to prove that $R(k, k) > \frac{\sqrt{2}}{\varepsilon}(1 + o(1))k2^{k/2}$. This is an improvement of a multiplicative factor of 2 over the naïve probabilistic bound in Erdős' 1947 paper.

7. Do the following:

• An amusing philosophical diversion is the St. Petersburg paradox. The game is simple. A coin is flipped until heads occurs. If heads occurs on the first flip, the house pays $2. If the first flip is tails and the second heads, the house pays $2^2 = 4. In general, if heads occurs first on the $i^{th}$ flip, then the house pays $2^i$. Would you pay $1000 to play this game? Why or why not?

• Write (and solve) a “good” homework problem based on weeks 3 through 7.

Good problems from HW1 (not eligible, just for your own enlightenment):

• Suppose that we have $3p$ points in the plane and that, for any two points $x$ and $y$, we have $\|x - y\|_2 \leq 1$. Show that at most $3p^2$ of the points are separated by a distance greater than $\sqrt{2}/2$.

• Use Ramsey’s theorem to show that for any $k, \ell \in \mathbb{N}$, there exists an $n$ such that every sequence of $n$ distinct integers contains an increasing subsequence of length $k + 1$ or a decreasing subsequence of length $\ell + 1$. (Ex. 9.6 in Diestel)