690I Scribe Notes for Feb 28 2006

Chad Brewbaker

April 5, 2006
0.1 scriven

Recall Erdős-Stone:
For any integers $p \geq 2$, $t \geq 1$,

$$\text{ex}(n, K_k(t, \ldots t)) = \left(1 - \frac{1}{p-1} * \binom{n}{2}\right) + o(n^2)$$

where $\text{ex}(n, G)$ is the maximum number of edges in an order $n$ graph with no subgraph isomorphic to $G$.

Erdős-Simonovits:
Let $L$ be a family of graphs, $p = \min\{\chi(l)\} 1, l \in L$. Let $\text{ex}(n, L) =$ the maximum number of edges in an order $n$ graph with no subgraphs isomorphic to any member $l \in L$. Then $\text{ex}(n, L) = \left(1 - \frac{1}{p-1}\right) \binom{n}{2} + o(n^2)$.

The proof is when $\max_{l \in L} (|V(l)|) \leq K$.

If you have a $K_p(K, \ldots K) \supseteq l$ in the original.

Let $\epsilon = \left(\frac{\beta}{6}\right)^h, h = |V(H)|$.

Then $e(G_n) > \left(1 - \frac{1}{p-1} + \beta\right) \binom{n}{2}$

$$\Rightarrow \|H \rightarrow G_n\| > \left(\frac{en}{M(\epsilon)}\right)^h$$

Recall Hajnal-Szemerédi
If $\delta(G_n) \geq (1 - \frac{1}{r}) n$, then there exists a subgraph which consists of $\left\lfloor \frac{n}{r} \right\rfloor$ vertex disjoint copies of $K_r$ for all $n$.

Let $C_4$ be the cycle on 4 vertices.

If $\delta(G_n) \geq \frac{3}{4} n$, then $\exists \left\lfloor \frac{n}{r} \right\rfloor$ vertex disjoint copies of $K_4$, hence $C_4$.

Equivalent to Hanjal-Szemerédi if $\Delta(G_n) < \frac{n}{2}$ and $r|n$, then there exists a proper coloring of $G_n$ where every color class is of size exactly $r$.

**Theorem 1** (Alon-Yuster(1992)). $\forall \alpha > 0$ and graph $H$, $\exists n_0$ such that in $n \geq n_0$, $\delta(G_n) > (1 - \frac{1}{\chi(H)} + \alpha)n$ then there is a family of $\lfloor \frac{1-\alpha}{\chi(H)} \rfloor n$ vertex-disjoint copies of $H$ in $G_n$. 

1
If $\alpha(G_n) > (\frac{1}{2} + \alpha)n$ then there is a family of $\lfloor \frac{(1-\alpha)n}{4} \rfloor$ vertex-disjoint copies of $C_4$ (for $n \geq n_0$).

**Conjecture 1.** $\forall H, \exists K$ such that if $\delta(G_n) > (1 - \frac{1}{\chi(H)})n$ then $G_n$ has a family of vertex-disjoint copies of $H$ that we use up all but $K$ vertices. ($k \neq 0$ even if $V(H)$ divides $n$)

Let $H = K_2, \delta(G_n) \geq \frac{n}{2}$

Dirac says $G_n \supseteq C_n$ a cycle on $n$ vertices.

If $n$ is even $C_n \supseteq$ a perfect matching.

If $n$ is odd $C_n \supseteq$ a matching of size $\lfloor \frac{n}{2} \rfloor$.

If $H = K_1$ then $K = 0$, if $H = K_2$ then $K = 1$.

$K \neq 0$ is shown by Alon-Yuster[1].

Tripartite $G = (V_1, V_2, V_3) | V_1 | = V_2 | = V_3 | = N$.

If each bipartite graph $(v_i, v_j)$ has minimum degree $\leq \frac{n}{3} + 2h - 1$, then there exists $\lfloor \frac{N}{h} \rfloor$ copies of $K_{h,h,h}$.

**Definition 1** ($(\epsilon, \delta)$-super-regularity). Given a pair $(A, B)$ we say that $(A, B)$ is $(\epsilon - \delta)$-super-regular if $\forall X \subseteq A, \forall Y \subseteq B$ satisfying $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we have $e(x, y) > \delta|X||Y|$ and $\deg(a) > \delta|B|\forall a \in A$

$\deg(b) > \delta|A|\forall b \in B$

**Theorem 2.** Let $\epsilon < \frac{1}{2}, d > 2\epsilon$. Let $(A, B)$ be an $\epsilon$-regular pair with density $d$, $|A| = |B| = L$.

Then $\exists A' \subseteq A, B' \subseteq B$ such that $(A', B')$ is $(\epsilon, d - 2\epsilon)$ super-regular, $|A'|, |B'| \geq (1 - \epsilon)L$

$d(A', B') \in (d - \epsilon, d + \epsilon)$

**Lemma 1** (Blow-up lemma by Komlós-Sárközy-Szemerédi(1994)). Given a graph $R$ of order $r$ and positive parameters, $\delta, \Delta$, there exists $\epsilon > 0$ such that the following holds.

Let $n_1, \ldots, n_r$ be arbitrary positive integers and replace the vertices of $R$ with pairwise disjoint sets $V_1, \ldots, V_r$ of sizes $n_1, \ldots, n_0$ (blowing up)

$R(n_1, \ldots, n_r)$ is obtained by replacing each edge $(v_i, v_j) \in E(R)$ with the complete bipartite graph between $V_i$ and $V_j$. 2
$G$ is obtained by replacing each edge $(v_i, v_j) \in E(R)$ with an $(\epsilon, \delta)$ super regular pair.

If a graph $H$, $\Delta(H) \leq \Delta$ (max degree) is embeddable into $R(n_1, \ldots, n_r)$ then it is embeddable into $G$.

For example let $\delta$ be given, then there exists $\epsilon > 0$ such that every $(\epsilon, \delta)$-super-regular pair (with both sets the same size) is Hamiltonian.

$R = K_2$
$R(L, L)$ is Hamiltonian
$G = (A, B), |A| = |B| = L$, $(\epsilon, \delta)$-super-regular

Every $\epsilon$-regular pair $|A| = |B| = L$ of density $\geq \delta + 2\epsilon$, $(\epsilon < \frac{1}{2})$, has a cycle of length $\geq 2(1 - \epsilon)L$. 


Bibliography