**Definition:** Let \((A, B; E)\) be a bipartite graph with density \(d\). We say that \((A, B; E)\) is \(\epsilon\)-regular if \(\forall X \subseteq A, \forall Y \subseteq B\) with \(|X| \geq \epsilon |A|, |Y| \geq \epsilon |B|\) we have \(|d(X, Y) - d| \leq \epsilon\).

**Proposition 1:** Let \(\epsilon < \epsilon'\). If \((A, B)\) is an \(\epsilon\)-regular pair, then \((A, B)\) is an \(\epsilon'\)-regular pair.

**Proof:** All we have done is relax the conditions.

**Example:** Let \((A, B)\) be a random bipartite graph with edge probability \(p\).

Fix \(0 < \epsilon \ll p\). Then

\[
\Pr (\exists a \in A : |\deg(a) - |B|| \geq \epsilon |B|) \leq |A| \Pr (|\deg(a) - |B|| \geq \epsilon |B|) = |A| \Pr \left( |\deg(a) - |B|| \geq \frac{\epsilon |B|}{\sqrt{|B|p(1-p)}} \sqrt{|B|p(1-p)} \right) \leq \exp \left( -\frac{1}{4} \left( \frac{\epsilon |B|}{\sqrt{|B|p(1-p)}} \right)^2 \right)
\]

And as our number of vertices in the graph \(n \to \infty\) our last term goes to 0.

**Proposition 2:** Let \((A, B)\) be an \(\epsilon\)-regular pair with density \(d\). Let \(A'\) be the set of vertices with \(\deg(a) \in [(d - \epsilon)|B|, (d + \epsilon)|B|]\). Then \(|A'| \geq (1 - 2\epsilon)|A|\).

**Proof:** Let \(X = \{a \in A : \deg(a) < (d - \epsilon)|B|\}, Y = B, \epsilon(X, Y) < (d - \epsilon)|Y||X|, d(X, Y) - d < -\epsilon\). Then either \(|X| < \epsilon |A|\) or \(|Y| < \epsilon |B|\). So \(|X| < \epsilon |A|\).

Similarly \(|\{a \in A : \deg(a) > (d + \epsilon)|B|\}| < \epsilon |A|\) so \(|A'| \geq |A| - 2\epsilon |A|\).

**Proposition 3:** (Intersection property) Let \((A, B)\) be \(\epsilon\)-regular with density \(d\) and \(0 < \epsilon \leq d < 1\). If \(Y \subseteq B\) and \((d - \epsilon)^{k-1}|Y| \geq \epsilon |B|\) for some \(k \geq 1\) then

\[
|\{(x_1, x_2, ..., x_k) : x_i \in A, |Y \cap \left( \bigcap_{i=1}^{k} N(x_i) \right)| < (d - \epsilon)^k|Y|\}| \leq k\epsilon |A|^k.
\]

**Proof:** By induction on \(k\).

Base case. Look at \(k = 1\). \(X = \{x_1 \in A : |y \cap N(x_1)| \leq (d - \epsilon)|Y|\}, |X| < \epsilon A\) otherwise
\(|X| \geq \epsilon |A|, |Y| \geq \epsilon |B|\) and \(d(X, Y) - d < -\epsilon\).

Inductive Hypothesis. Assume true for \(k - 1 (k \geq 2)\).

Let \((A, B)\) be an \(\epsilon\)-regular pair with density \(d\) and \(Y \subseteq B, (d - \epsilon)^{k-1}|Y| \geq \epsilon |B|\).

Let \(S_1 = \{(x_1, x_2, \ldots, x_{k-1}, x_k) : x_i \in A, Y \cap \left( \bigcap_{i=1}^{k-1} N(x_i) \right) \leq (d - \epsilon)^{k-1}|Y| \}\).

Certainly \((d - \epsilon)^{k-2}|Y| \geq \epsilon |B|\). By the inductive hypothesis \(|S_1| \leq (k - 1)\epsilon |A|^{k-1}|A|\).

Let \(S_2 = \{(x_1, x_2, \ldots, x_{k-1}, x_k) : x_i \in A, Y \cap \left( \bigcap_{i=1}^{k-1} N(x_i) \right) < (d - \epsilon)^k|Y|, \bigcap_{i=1}^{k-1} N(x_i) \geq (d - \epsilon)^{k-1}|Y| \}\).

Let \(Y' = Y \cap \left( \bigcap_{i=1}^{k-1} N(x_i) \right)\).

Then \(|Y'| \geq (d - \epsilon)^{k-1}|Y| \geq \epsilon |B|\) (due to the hypothesis assumption).

By the base case \(|\{x_k \in A : Y' \cap N(x_k) < (d - \epsilon)|Y'|\}| < \epsilon |A|\).

Because \((d - \epsilon)^k|Y| \leq (d - \epsilon)|Y'|\) this set contains \(\{x_k \in A : Y \cap N(x_k) < (d - \epsilon)^k|Y|\}\).

So \(|S_2| \leq |A|^k\) and \(|S_1| + |S_2| \leq k\epsilon |A|^k\). \(\blacksquare\)

**PROPOSITION 4:** Let \(A'(1 - k\epsilon) > 1, (d - \epsilon)^k|B| \geq l, (d - \epsilon)^{k-1} \geq \epsilon\). Then any \(\epsilon\)-regular pair with density \(d\) contains a \(K_{k,l}\).

**PROOF:** \[\left| \{(x_1, x_2, \ldots, x_k) \in A^k : \bigcap_{i=1}^n N(x_i) \geq (d - \epsilon)^k|B| \} \right| > (1 - k\epsilon)|A|^k > 1.\]

So \(\exists (x_1, x_2, \ldots, x_k)\) such that \(\bigcap_{i=1}^k N(x_i) \geq (d - \epsilon)k|B| \geq l\) (as long as \((d - \epsilon)^{k-1}|B| \geq \epsilon |B|\)). \(\blacksquare\)

**HOMWORK:** (Counts as 2) Prove a general version of the intersection property. Find \(f = f(d, \epsilon, k, l)\). Let \(k, l\) be integers with the property that \(\forall Y \subseteq B\) if \(|Y| \geq f|B|\) then
\[\left| \{(x_1, x_2, \ldots, x_k) \in A^k : Y \cap \left( \bigcap_{i=1}^l N(x_i) \cap \bigcap_{i=l+1}^k N(x_i) \right) < (d - \epsilon)^l(1 - d - \epsilon)^{k-l}|Y| \} \right| \leq k\epsilon |A|^k.\]
\[ N(x_i) = B \setminus N(x_i). \]

**Homework:** Prove I.P. with \( < (d - \epsilon)^k |Y| \) replaced by \( > (d + \epsilon)^k |Y| \). You will need a \( |Y| \geq f(d, \epsilon, k)|B| \) condition.

**Example:** Let \( H \) be a bipartite graph on \( k \times l \) vertices. Then for \( 0 < \epsilon < d < 1 \) if \( |A| = |B| = N \) is large enough, an \( \epsilon \)-regular pair \( (A, B) \) contains \( H \) as an induced subgraph.