M606 - Posets - Dilworth's Theorem

Note Title 3/22/2007

Definition A chain cover of a poset is a family of chains whose union is the poset.

Theorem [Dilworth, 1950]
If P is a finite poset, then the maximum size of an antichain in P is the minimum size of a chain cover.

Proof [Perles, 1963]

Since no chain contains two elements of an antichain, $w(P)$ is at most the size of a chain cover.

To show the reverse, induction on $|P|$. $|P|=0$ is trivial.

Let $|P|\geq 1$ and A be

a maximum antichain in $\mathcal{P}$

<u>Case 1</u> A excludes some minimal element of $\mathcal{P}$ and some maximal element of $\mathcal{P}$.

Apply the IH to $D[A]$ and $U[A]$. Let $|A| = k$.

There are k chains that cover $U[A]$ and k that cover $D[A]$. Let $u_1 > \cdots > u_m = a$ be one in $U[A]$ and $a = v_n > \cdots > v_1$ be one in $D[A]$.

Then $u_1 > \cdots > u_m = a = v_n > \cdots > v_1$ is a chain in $\mathcal{P}$. "Glue" all such chains together and we obtain k chains that cover $\mathcal{P}$.

<u>Case 2</u> A contains all maximal or all minimal elements

If both, then P is antichain A, and the proof is finished, trivially.

WLOG, assume A contains all maximal elements. This implies A is the set of maximal elements.

Moreover, $\exists$ a maximal M that is also not minimal, so there is a minimal $m < M$.

Since every maximum antichain is either maximal or minimal elements,

$P - \{m, M\}$ removes exactly one element from every k-element antichain.
So $\omega(P - \{m, M\}) \leq k-1$.

Hence, there are $k-1$ chains that cover $P - \{m, M\}$. Add $\{m, M\}$ as a 2-element chain and the theorem is proven. $\square$

HW: Prove that Dilworth's Theorem on posets is equivalent to the König–Hall matching theorem on bipartite graphs.

Lots of hints.

Thanks to Doug West for his course notes.