M606 - Generating Functions - Unimodality

Note Title 2/27/2007

A sequence c_n is unimodal if there are indices $r \le s$ such that

$$c_0 \le c_1 \le \cdots \le c_r = c_{r+1} = \cdots = c_{s-1} = c_s \ge c_{s+1} \ge \cdots \ge c_n$$

For $c_k = \binom{n}{k}$ $\quad r = \lfloor n/2 \rfloor, \quad s = \lceil n/2 \rceil$

The function f is <u>concave</u> if

$$f\left(\frac{x+y}{2}\right) \ge \frac{f(x)+f(y)}{2}$$

The function f is <u>log concave</u> if

$$\ln f\left(\frac{x+y}{2}\right) \ge \frac{\ln f(x) + \ln f(y)}{2}$$

$$f\left(\frac{x+y}{2}\right) \ge \sqrt{f(x)f(y)}$$

$$\left[f\left(\frac{x+y}{2}\right)\right]^2 \ge f(x)f(y)$$

A sequence $\{c_k\}$ is log concave if

$$c_k^2 \ge c_{k-1}c_{k+1}$$

Prop positive + log concave $\Rightarrow$ unimodal

Proof If not unimodal, then $\exists\, k$ s.t.

$$C_{k-1} > C_k < C_{k+1}.$$

But in that case, $C_k^2 < C_{k-1} C_{k+1}$. $\square$

Example Binomial coefficients

$$\frac{\binom{n}{k}^2}{\binom{n}{k+1}\binom{n}{k-1}} = \frac{\dfrac{n!^2}{k!^2 (n-k)!^2}}{\dfrac{n!^2}{(k-1)!(k+1)!(n-k+1)!(n-k-1)!}}$$

$$= \frac{(k+1)!(k-1)!\,(n-k+1)!\,(n-k-1)!}{k!\;\;k!\;\;(n-k)!\;\;(n-k)!}$$

$$= \frac{k+1}{k}\,\frac{n-k+1}{n-k} > 1$$

So $\left\{\binom{n}{k}\right\}_k$ is log concave $\Rightarrow$ unimodal

<u>Theorem</u> Let $p(x) = c_0 + c_1 x + \cdots + c_n x^n$ be a polynomial all of whose zeros are real and negative. Then $\{c_k\}_{k=0}^n$ is strictly log concave.

<u>Proof</u>

<u>Lemma</u>
Let $f(x,y) = c_0 x^n + c_1 x^{n-1} y + \cdots + c_n y^n$ all of whose roots y/x are real. Let g be a derivative with respect to x's and y's. Then all of g's zeros are real

<u>Proof</u> $g f_0$ + {zeros of f' real
(Rolle's thm.) if zeros of f real} $\square$

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n = \prod_{j=1}^{n} (x + x_j)$$

Apply $D_x^m D_y^{n-m-2}$ to f. The following survives:

$$\frac{c_{n-m-2}\,(m+2)}{n-m-1} x^2 + 2 c_{n-m-1} xy + \frac{(n-m) c_{n-m}}{m+1} y^2$$

Let $c_j = \binom{n}{j} d_j$.

We now have

$$\binom{n}{m+1}\left[d_{n-m-2}x^2+2d_{n-m-1}xy+d_{n-m}y^2\right]$$

This is quadratic, must have 2 real roots, hence

$$(2d_{n-m-1})^2-4d_{n-m-2}d_{n-m}\geq 0$$

$$\Rightarrow \quad d_{n-m-1}^2 \geq d_{n-m-2}d_{n-m}$$

$$\frac{c_{n-m-1}^2}{\binom{n}{n-m-1}^2} \geq \frac{c_{n-m-2}c_{n-m}}{\binom{n}{n-m-2}\binom{n}{n-m}}$$

$$c_{n-m-1}^2 \geq \frac{(m+2)(n-m)}{(m+1)(n-m-1)}c_{n-m-2}c_{n-m}$$

$$> c_{n-m-2}c_{n-m}$$

$\Rightarrow$ log concavity $\Rightarrow$ unimodality. $\square$

<u>Example</u> Binomial coefficients

$$\sum_{k=0}^{n}\binom{n}{k}x^k = (1+x)^n \qquad (\text{all roots } = -1)$$

Example signless stirling 1^{st} kind

$$\sum_k c(n,k)\, x^k = (x+n-1)_n$$
$$= x(x+1)\cdots(x+n-1)$$

all roots $= \{0, -1, \ldots, -(n-1)\}$

Since $c(n,0) = 0$ $\forall\, n \geq 1$, then

$$\sum_k c(n,k)\, x^{k-1} = (x+1)\cdots(x+n-1)$$

So, $\{c(n,k)\}_{k=1}^{n}$ is log concave, hence unimodal, when $n \geq 1$.

Example Stirling numbers of the second kind

$$A_n(x) = \sum_j S(n,j)\, x^j$$

Fact: zeros are hard to find.
Fact: they are real and negative

Recall:

$$\begin{cases} A_n(y) = (y + y D_y) A_{n-1}(y) & n \geq 1 \\ A_0 \equiv 1 \end{cases}$$

Or: $e^y A_n(y) = y \left(e^y A_{n-1}(y)\right)'$

<u>Claim</u> $e^y A_n(y)$ has exactly n distinct zeros, all of which are negative (except $x=0$)

<u>POC</u> Induction on n

$n=0$ ✓

Suppose the claim is true for $0, 1, \ldots, n-1$.

By Rolle's theorem (if $f(a)=f(b)$, $\exists$ $c \in [a,b]$ with $f'(c)=0$),

$\left(e^y A_{n-1}(y)\right)'$ has $n-2$ distinct zeros, all of them negative. So $y\left(e^y A_{n-1}(y)\right)'$ has $n-1$. Only 0 is non-negative.

Let them be $z_{n-1} \le z_{n-2} \le \cdots \le z_1 = 0$.

But $\lim_{y \to -\infty} e^y A_{n-1}(y) = 0$

because $A_{n-1}(y)$ is a polynomial.

So, $\left(e^y A_{n-1}(y)\right)' = 0$ for some $z_n < z_{n-1}$. $\square$

Not necessary.

If $b(n,k)$ denotes the number of permutations with exactly k inversions, then $\{b(n,k)\}_{k\ge 0}$ is unimodal, but

$$\{b(n,k)\}_{k\ge 0} \xleftrightarrow{\text{ops}} (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1})$$

which has many nonreal roots.

In fact, $b(n,k)$ increases for $k \le \frac{1}{2}\binom{n}{2}$ and then decreases.